18(3): Higher Order Corrections to the Lamb Shift.

From previous notes, the Lamb shift is worked out from: - (1)

$$\langle \Delta U \rangle = \langle \Delta U \rangle^{(2)} + \langle \Delta U \rangle^{(4)} + \langle \Delta U \rangle^{(6)} + \dots$$

where U is the Coulomb potential between the electron and proton of the H atom:

$$U = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(2)$$

In Cartesian coordinates:

$$r = (x^2 + y^2 + z^2)^{1/2} \quad -(3)$$

In eq. (1):

$$\langle \Delta U \rangle^{(2)} = \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right).$$

$$= \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^2 \left(-\frac{e^2}{4\pi\epsilon_0 r} \right) \quad -(4)$$

On the classical level:

$$\langle \Delta U \rangle^{(2)} = 0 \quad -(5)$$

because

$$\nabla^2 \left(\frac{1}{r} \right) = 0 \quad -(6)$$

from vector algebra. However on the quantum level of the Lamb shift calculation:

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi \delta_D \quad -(7)$$

where δ_D is the Dirac delta function. The expectation value is used in the calculation:

$$\left\langle \nabla^2 \left(-\frac{e^2}{4\pi\epsilon_0 r} \right) \right\rangle = \frac{e^2}{\epsilon_0} \int \psi^* \delta_D \psi \, d\tau \quad -(8)$$

) where $\int \phi^* \delta_D \phi \, d\tau = |\phi(0)|^2$ — (9)

However, $\langle \Delta U \rangle^{(4)}$ and $\langle \Delta U \rangle^{(6)}$ exist on the classical level.

They can be computed with computer algebra and are given by:

$$\langle \Delta U \rangle^{(4)} = \frac{1}{216} \langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle \left(\frac{\partial^4 U}{\partial X^4} + \frac{\partial^4 U}{\partial Y^4} + \frac{\partial^4 U}{\partial Z^4} + 6 \left(\frac{\partial^4 U}{\partial Y^2 \partial Z^2} + \frac{\partial^4 U}{\partial X^2 \partial Z^2} + \frac{\partial^4 U}{\partial X^2 \partial Y^2} \right) \right) \quad -(10)$$

and

$$\langle \Delta U \rangle^{(6)} = \frac{\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle}{19440} \left(\frac{\partial^6 U}{\partial X^6} + \frac{\partial^6 U}{\partial Y^6} + \frac{\partial^6 U}{\partial Z^6} + 15 \left(\frac{\partial^6 U}{\partial Y^4 \partial Z^2} + \frac{\partial^6 U}{\partial Y^2 \partial Z^4} + \frac{\partial^6 U}{\partial X^4 \partial Z^2} + \frac{\partial^6 U}{\partial X^4 \partial Y^2} + \frac{\partial^6 U}{\partial X^2 \partial Z^4} + \frac{\partial^6 U}{\partial X^2 \partial Y^4} \right) + 90 \frac{\partial^6 U}{\partial X^2 \partial Y^2 \partial Z^2} \right) \quad -(11)$$

As in the previous note:

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{2}{\pi} \alpha \lambda^2 \int_{\pi/a_0}^{mc/\hbar} \frac{d\kappa}{\kappa} \quad -(12)$$

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle = \frac{4}{V} (\alpha \lambda^2)^2 \int_{\pi/a_0}^{mc/\hbar} \frac{d\kappa}{\kappa^4} \quad -(13)$$

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle = \frac{8\pi}{V^2} (\alpha \lambda^2)^3 \int_{\pi/a_0}^{mc/\hbar} \frac{d\kappa}{\kappa^7} \quad -(14)$$

where the fine structure constant is:

$$\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0} \quad -(15)$$

and

$$\lambda = \frac{\hbar}{mc}. \quad -(16)$$

where V is the volume of radiation.

It is claimed that the Lamb shift is given by

$$\langle \Delta U \rangle^{(2)} = \frac{1}{6} \langle \underline{\delta r} \cdot \underline{\delta r} \rangle \frac{e^2}{\epsilon_0} |\phi(0)|^2 \quad -(17)$$

where

$$\langle \underline{\delta r} \cdot \underline{\delta r} \rangle = \frac{2}{\pi} \alpha \lambda^2 \log_e \frac{1}{\pi\alpha} \quad -(18)$$

$$= \frac{2}{\pi} \alpha \left(\frac{\hbar}{mc}\right)^2 \log_e \frac{1}{\pi\alpha}$$

$$= \text{constant}$$

in which m is the mass of the electron. So $\langle \underline{\delta r} \cdot \underline{\delta r} \rangle$ is the fluctuation in the position of the electron due to the vacuum. In eq. (17):

$$\langle \nabla^2 U \rangle = \frac{e^2}{\epsilon_0} |\phi(0)|^2 \quad -(19)$$

where $\phi(0)$ is the value of the relevant H atom wave function at the nucleus. The units on the left hand side of eq. (19) are Jm^{-2}, and $e^2/\epsilon_0 = Jm$, so the units of $|\phi(0)|^2$ must be m^{-3}. This is consistent with the fact that the units of the Dirac delta function in in Eq. (9) is n^{-3}.

For the 2S level of the H atom:

$$|\phi_{2S}(0)|^2 = \frac{1}{8\pi a_0^3} \quad -(20)$$

where a_0 is the Bohr radius:

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} = \alpha\lambda \quad -(21)$$

So

$$\langle \nabla^2 U(2S) \rangle = \frac{e^2}{8\pi\epsilon_0 \alpha^3\lambda^3} \quad -(22)$$

$$= \frac{1}{2}\frac{\hbar c}{\alpha^2\lambda^3}$$

It follows that:

$$\langle \Delta U \rangle^{(2)} = \frac{1}{6}\left(\frac{2}{\pi}\alpha\lambda^2\right)\log_e \frac{1}{\pi\alpha} \cdot \frac{\hbar c}{2\alpha^2\lambda^3}$$

$$= \frac{1}{6\pi}\cdot\frac{\hbar c}{\alpha\lambda}\log_e\frac{1}{\pi\alpha} \quad -(23)$$

$$\boxed{\langle \Delta U \rangle^{(2)} = \frac{\hbar c}{6\pi a_0}\log_e\frac{1}{\pi\alpha}} \quad -(24)$$

which is a universal constant and the conversion units of J.

Therefore in eq. (1):

$$\langle \Delta U \rangle = \frac{\hbar c}{6\pi a_0}\log_e\frac{1}{\pi\alpha} + \langle \Delta U \rangle^{(4)} + \langle \Delta U \rangle^{(6)} + \dots \quad -(25)$$

) where $\langle \Delta U \rangle^{(4)}$ and $\langle \Delta U \rangle^{(6)}$ exist on a classical level and from eqs. (13) and (14) depend inversely on the radiation volume $\bar{V}$.

Denoting:

$$\langle \Delta \bar{U} \rangle^{(4)} = \frac{1}{216} \langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle f_1(X,Y,Z) \quad -(26)$$

and

$$\langle \Delta \bar{U} \rangle^{(6)} = \frac{\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle}{19440} f_2(X,Y,Z) \quad -(27)$$

the expectation values:

$$\langle f_1(X,Y,Z) \rangle = \int \phi^* f_1(X,Y,Z) \phi \, d\tau \quad -(28)$$

and

$$\langle f_2(X,Y,Z) \rangle = \int \phi^* f_2(X,Y,Z) \phi \, d\tau \quad -(29)$$

can be computed for any H atom wavefunction. In order to do this the functions have to be transformed from Cartesian to spherical polar coordinates.

However, the complexity of this procedure can be eliminated by realizing that:

$$f_1(X,Y,Z) = \langle f_1(X,Y,Z) \rangle \quad -(30)$$

and

$$f_2(X,Y,Z) = \langle f_2(X,Y,Z) \rangle \quad -(31)$$

i.e. that f_1 and f_2 are already classical or expectation values.

So higher order terms introduce extra detail into the Lamb shift.