15(1): Calculation of the Mean Square of the Fluctuating Spin Connection.

In this calculation the magnitude of the fluctuating position vector is denoted:

$$r = r_0 + \delta r \quad -(1)$$

so

$$\underline{r} = (r_0 + \delta r)\,\underline{e}_r \quad -(2)$$

In electrostatics the total scalar potential is:

$$\phi = -\frac{e}{4\pi\epsilon_0 r} = -\frac{e}{4\pi\epsilon_0 (r_0 + \delta r)} \quad -(3)$$

The electric field strength is denoted:

$$\underline{E} = -\underline{\nabla}\phi = -\frac{\partial\phi}{\partial r} = -\frac{e}{4\pi\epsilon_0 r^2}\,\underline{e}_r$$

$$= \frac{-e}{4\pi\epsilon_0 (r+\delta r)^2}\,\underline{e}_r$$

$$= -\underline{\nabla}\phi_0 + \underline{\omega}\,\phi_0 \quad -(4)$$

where

$$\phi_0 = -\frac{e}{4\pi\epsilon_0 r_0} \quad -(5)$$

is the potential in the hypothetical absence of the vacuum.

It follows that:

$$\frac{-e}{4\pi\epsilon_0 (r_0+\delta r)^2}\,\underline{e}_r = -\frac{e}{4\pi\epsilon_0 r_0^2}\,\underline{e}_r - \frac{e}{4\pi\epsilon_0 r_0}\,\underline{\omega} \quad -(6)$$

so

$$\frac{1}{(r+\delta r)^2}\,\underline{e}_r = \frac{1}{r^2}\,\underline{e}_r + \frac{1}{r}\,\underline{\omega} \quad -(7)$$

so

$$\underline{\omega} = \left(\frac{1}{(r+\delta r)^2} - \frac{1}{r^2}\right) r\,\underline{e}_r \quad -(8)$$

where $$\underline{r} = r\,\underline{e}_r \quad (9)$$

Eq. (8) can be expressed as:

$$\underline{\omega} = \frac{1}{r}\left(\frac{1}{(1+x)^2} - 1\right)\underline{e}_r \quad (10)$$

where

$$x = \frac{\delta r}{r} \quad (11)$$

Using the binomial series:

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots \quad (12)$$

for

$$x << 1 \quad (13)$$

it is found that the mean of the spin connection vector is:

$$\boxed{\langle \underline{\omega} \rangle = 3\frac{\langle \delta r^2 \rangle \underline{e}_r}{r^3}} \quad (14)$$

where we have used the isotropic condition:

$$\langle \delta \underline{r} \rangle = \underline{0} \quad (15)$$

It becomes clear that the ensemble averaged $\langle \delta r^2 \rangle$ of the vacuum spin connection is given by $\langle \delta r^2 \rangle$ of the vacuum. For an electron in an atom, it becomes clear that $\langle \omega \rangle$ is given by the expectation value of $\langle \delta r^2 \rangle$. These ideas can be developed for electrodynamics in general.