The classical equation of the relativistic electron is:
\[ E^2 = p^2 c^2 + m^2 c^4 \quad -(1) \]
and the relativistic quantization is:
\[ E \psi = i \hbar \frac{\partial \psi}{\partial t}, \quad -(2) \]
\[ p \psi = -i \hbar \frac{\partial \psi}{\partial \mathbf{x}} \quad -(3) \]
\[ \therefore \quad \frac{d}{dt} \left[ -\frac{\hbar^2}{2m} \nabla^2 \psi \right] + m^2 c^4 \psi \psi = 0 \quad -(4) \]
\[ i.e. \quad \left( \frac{\partial^2}{\partial t^2} + \frac{(mc)^2}{\hbar^2} \right) \psi = 0 \quad -(5) \]
which is a limit of the ECE wave equation obtained from the tetrad postulate. The d'Alembertian is:
\[ \Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \quad -(6) \]
The Compton wavelength is:
\[ \lambda_c = \frac{mc}{\hbar} \quad -(7) \]
The de Broglie-Einstein relativity follow:
\[ E = \hbar \omega = \hbar mc^2, \quad p = \hbar k = \hbar k_0 \quad -(8) \]
A solution of eq. (5) is:
\[ \psi \propto \exp \left( -i(\omega t - k x) \right) \quad (9) \]

For Eqs. (5) and (9):
\[
\left( -\frac{\alpha^2}{c^2} + ic^2 \right) \psi = -\left( \frac{mc}{\hbar} \right)^2 \psi \quad (10)
\]

i.e.
\[
\frac{\alpha^2}{c^2} = \kappa^2 + m^2 c^2 \quad (11)
\]

so.
\[
\frac{\beta^2}{c^2} = \frac{\kappa^2}{c^2} \kappa^2 + m^2 c^4 \quad (12)
\]

Eqs. (1), (8) and (12) are self consistent, Q.E.D.

Eq. (1) can be rewritten as:
\[
E = mc^2 = \frac{\beta c}{\sqrt{1 + \frac{\beta^2}{c^2}}} = \frac{p c}{\sqrt{E^2 + mc^2}} \quad (13)
\]

so the relativistic 
\[
T = \left( 1 - \frac{\beta}{c} \right) mc^2 = \frac{p}{m \sqrt{1 + \frac{\beta^2}{c^2}}} \quad (14)
\]

In the classical non-relativistic limit:
\[
\beta \ll c \quad (15)
\]

so
\[
T \approx \frac{1}{2} m \nu^2 = \frac{p^2}{2m} \quad (16)
\]

Q.E.D.

In \( \mathfrak{g}_\mathbb{C} \), \( \mathfrak{su}(2) \) basis:
\[ E - mc^2 = \frac{\sigma \cdot P - \sigma \cdot P}{m} \frac{1}{(1 + \gamma)} E + mc^2 \] - (17)

The interaction of the electron beam with the ECE vacuum is described by the relativistic minimal prescription:

\[ p^\mu = p - eA_{\text{vac}}^\mu \] - (18)

where:

\[ p^\mu = \left( \frac{E}{c}, \vec{p} \right), \quad A_{\text{vac}}^\mu = \left( \phi_{\text{vac}}, \vec{A}_{\text{vac}} \right) \] - (19)

Therefore:

\[ E \to E - e\phi_{\text{vac}}, \quad \vec{p} \to \vec{p} - e\vec{A}_{\text{vac}} \] - (20, 21)

If, in addition, an external magnetic field is applied, it is defined conventionally as:

\[ \vec{B} = \nabla \times \vec{A} \] - (22)

Then:

\[ E \to E - e\phi_{\text{vac}}, \quad \vec{p} \to \vec{p} - e(\vec{A} + \vec{A}_{\text{vac}}) \] - (23, 24)

From the de Broglie / Einstein equations:

\[ \gamma = \frac{E}{mc^2} \] - (25)

In the usual assumption the quantization of Eq. (17) takes place as follows:
\[ (E - mc^2) \psi = \frac{1}{m} \left( \sigma \cdot (-i \mathbf{\nabla}) \left( \frac{1}{1 + \gamma} \sigma \cdot \mathbf{p} \psi \right) \right) \]  

Now note that:  
\[ \nabla \left( \frac{\mathbf{p} \psi}{m c} \right) = 0 \quad \text{(27)} \]

so  
\[ \nabla \left( \frac{1}{1 + \gamma} \right) = 0 \quad \text{(28)} \]

It follows that:  
\[ (E - mc^2) \psi = -\frac{i \hbar}{m(1 + \gamma)} \sigma \cdot \nabla \left( \sigma \cdot \mathbf{p} \psi \right) \]

\[ = -\frac{i \hbar}{m(1 + \gamma)} \left( \frac{\sigma \cdot \nabla \sigma \cdot \mathbf{p}}{m} \right) \psi + \left( \frac{\sigma \cdot \nabla \psi}{m} \right) \sigma \cdot \mathbf{p} \]  

where  
\[ \psi = \psi_0 \exp \left( -i (\omega t - \mathbf{k} \cdot \mathbf{z}) \right) \quad \text{(30)} \]

and  
\[ \mathbf{p} = \gamma \mathbf{p}_0 \quad \text{(31)} \]

Therefore:  
\[ \frac{\nabla}{\partial \mathbf{z}} \psi = \frac{\partial \psi}{\partial \mathbf{z}} \mathbf{k} \]

\[ = i k \mathbf{z} \cdot \mathbf{k} \]
The real part of eq. (29) is therefore:

\[ \Re \left( E - mc^2 \right) \psi = \frac{\hbar c}{m(1+Y)} \left( \sigma \cdot \mathbf{k} - \mathbf{c} \cdot \mathbf{p} \right) \psi - (33) \]

The expectation value of \( E \), equation (34), is:

\[ E - mc^2 = \frac{\hbar c \mathbf{p} \cdot \mathbf{k}}{m(1+Y)} = \frac{\hbar c \mathbf{p}_2}{m(1+Y)} - (34) \]

Using:

\[ p_2 = \frac{\hbar c}{m(1+Y)} \]

and

\[ m(1+Y) = \frac{(E + mc^2)}{c^2} - (36) \]

eq. (34) becomes, self-consistently:

\[ E - mc^2 = \frac{c^2 \mathbf{p}_2^2}{E + mc^2} - (37) \]

Q.E.D.

The imaginary part of Eq. (29) is:

\[ \Im \left( E - mc^2 \right) \psi = -i \frac{\hbar c}{m(1+Y)} \left( \sigma \cdot \nabla \sigma \cdot \mathbf{p} \right) \psi - (38) \]

\[ \sigma \cdot \nabla \sigma \cdot \mathbf{p} = \nabla \cdot \mathbf{p} + i \sigma \cdot \nabla \times \mathbf{p} \]

This gives a real part:

\[ \left( E - mc^2 \right) \psi = \frac{\hbar c}{m(1+Y)} \sigma \cdot \nabla \times \mathbf{p} \psi - (40) \]

where \( \sigma \cdot \nabla \times \mathbf{p} \) is the spin angular momentum of the electron.
\[
\sum = \frac{\hbar}{2} \sigma \quad -(41)
\]

For an electron moving in \(\mathbb{Z}\):
\[
p = p_z \mathbb{Z} \quad -(42)
\]

so
\[
\nabla \times p = 0 \quad -(43)
\]

However, if an external magnetic field \(B\) is applied to the relativistic electron beam:
\[
p \rightarrow p - eA \quad -(44)
\]

and
\[
(E - mc^2) \psi = -\frac{e\hbar}{m(1+y)} \sigma \cdot \nabla \times A \psi \quad -(45)
\]

This term gives rise to electron spin resonance in the relativistic electron beam as follows:
\[
(E - mc^2) \psi = -\frac{2e}{m(1+y)} \frac{\sum \cdot B}{\psi} \quad -(46)
\]

where
\[
y = \frac{\hbar c}{2mc^2} \quad -(47)
\]

and \(\omega\) is the angular frequency of the electron wave
For an external magnetic field aligned with some direction, \( Z \), \( s \) and \( \alpha \) in the Hamiltonian:

\[
(E - mc^2)\alpha = -\frac{2eBz}{m} \left( 1 + \frac{\hbar \omega}{mc^2} \right) S_z \alpha - (48)
\]

\[
= -\frac{2eBz}{m} S_z \alpha
\]

where \( m_s = \pm \frac{1}{2} \).

The ESR resonance frequency is:

\[
\omega_{ESR} = \frac{2eBz}{m} \left( 1 + \frac{\hbar \omega}{mc^2} \right) - (49)
\]

In the non-relativistic limit:

\[
\frac{\hbar \omega}{mc^2} \rightarrow mc^2 - (50)
\]

So

\[
\omega_{ESR} \rightarrow \frac{eBz}{m} - (51)
\]

This is the usual result, Q.E.D.

Eq. (49) is a rigorous test of relativistic quantum mechanics and the de Broglie/Einstein equations. It is also a rigorous test of the quantization condition:
\[ p^\mu = iv^\mu \quad \text{(52)} \]

It is implicitly assumed but rarely made clear that a relativistic four-vector \( A^\mu \) goes to a four-vector \( A^\mu \text{ vac} \). This is an axion, or assumption, of quantum mechanics.

**The Effect of \( \mathbf{A} \) Vacuum Potential**

The effect of \( \mathbf{A} \) Vacuum Potential: This is to change eq. (45) to:

\[ (E - mc^2) \psi = - \frac{e}{m(1 + \gamma)} \mathbf{A} \cdot \mathbf{v} \times (A_\text{vac} + A) \]

\[ \psi \]

\[ (E - mc^2) \psi = - \frac{2e}{m(1 + \gamma)} \mathbf{A} \cdot \mathbf{v} \times (A_\text{vac} + A) \]

So:

\[ (E - mc^2) \psi = - \frac{2e}{m(1 + \gamma)} \mathbf{A} \cdot \mathbf{v} \times (A_\text{vac} + A) \]

and the ESR resonance frequency is:

\[ \omega_{\text{ESR}} = \frac{2e}{m\left(1 + \frac{2e}{mc^2}\right)} \left( b_2 + \nabla \times A \right)_z \quad \text{(55)} \]

So the vacuum potential has no effect on the ESR resonance frequency. This demotes the existence of energy for space."