324(1): The Relativistic Binet Equation.

First consider the relativistic Lagrangean defined by:

\[ p = \frac{dL}{dV} \quad (1) \]

where \( p \) is the relativistic momentum:

\[ p = \gamma m v \quad (2) \]

As in Messiah and Thiria on page 539, the relativistic Lagrangean is:

\[ L = -mc^2 \frac{1}{v} - U \quad (3) \]

\[ = -mc^2 \left( 1 - \frac{v^2}{c^2} \right)^{1/2} - U \]

The relativistic Hamiltonian is defined by:

\[ H = \gamma p - L \]

\[ = \gamma m v^2 - L \]

\[ = \frac{p}{\gamma m} + mc^2 + U \quad (4) \]

where

\[ E = \gamma mc^2 \quad (5) \]

Now consider the Euler Lagrange equations with the relativistic Lagrangean (3):
1) \[ \frac{dL}{d\theta} = \frac{d}{dt} \frac{dL}{d\theta} = 0 \quad - (6) \]

and

\[ \frac{dL}{d\tau} = \frac{d}{dt} \frac{dL}{d\tau} = 0 \quad - (7) \]

In planar polar coordinates the velocity is defined by:

\[ v = r \dot{\theta} + \omega r \cos \theta \quad - (8) \]

\[ v^2 = \dot{r}^2 + \omega^2 r^2 \quad - (9) \]

so

The Lorentz factor is therefore:

\[ \gamma = \left( 1 - \frac{1}{c^2} (\dot{r}^2 + \dot{\theta}^2 r^2) \right)^{-1/2} \quad - (10) \]

Therefore:

\[ L = -mc^2 \left( 1 - \frac{1}{c^2} (\dot{r}^2 + \dot{\theta}^2 r^2) \right)^{1/2} - U \quad - (11) \]

For a central inverse square attraction between masses \( m \) and \( M \):

\[ U = -\frac{MB}{r} \quad - (12) \]

so

\[ L = -mc^2 \gamma^{1/2} + \frac{MB}{r} \quad - (13) \]

where:
\[ f = 1 - \frac{1}{c^2} (\dot{r}^2 + \dot{\theta}^2 + \dot{z}^2) \] - (14)

Therefore:

\[ L = \frac{dL}{d\theta} = \gamma m r^2 \dot{\theta} \] - (15)

and

\[ \frac{dL}{dt} = 0 \] - (16)

Therefore \( L \) is the conserved relativistic angular momentum.

From eq. (7):

\[ \delta m (\ddot{r} - \dot{\theta}^2) = F(r) = -\frac{dU}{dr} \] - (17)

Eq. (17) is to be compared with the Lorentz force equation in ECE 2 general relativity:

\[ F = \gamma m \left( a + \nu \times \Omega \right) - \frac{\nu^2}{1 + \frac{\nu \cdot q}{c}} \left( \frac{\nu \cdot q}{c} \right) \] - (18)

where \( \nu \) is the Newtonian acceleration:

\[ \frac{a}{N} = \frac{d^2 r}{dt^2} - \frac{r}{r} = \ddot{r} - (19) \]
Eq. (18) is more general than eq. (17) because the latter has assumed circular motion in a plane, whereas eq. (18) is true for general dynamics. However, both equations are equations of general relativity.

Eqs. (15) and (17) can be rewritten as the generally covariant bind equation for plane motion:

Consider:
\[
\frac{d}{dt} \left( \frac{1}{r} \right) = - \frac{1}{r^2} \frac{dr}{dt} = - \frac{1}{r^2} \frac{dr}{dt} \frac{dt}{d\theta} - (20)
\]
where
\[
\frac{d\theta}{dt} = \frac{L}{\gamma m r^2} - (21)
\]
From eq. (15): So:
\[
\frac{d}{d\theta} \left( \frac{1}{r} \right) = - \frac{\gamma m}{L} \frac{dr}{dt} - (22)
\]
Similarly:
\[
\frac{d^2}{d\theta^2} \left( \frac{1}{r} \right) = - \frac{\gamma m}{L} \frac{dr}{dt} \frac{d^2r}{dt^2}
\]
\[
= - \frac{\gamma m}{L} \frac{dt}{d\theta} \frac{d^2r}{dt^2}
\]
\[
= - \frac{\gamma m r^2}{L^2} \frac{d^2r}{dt^2} - (23)
\]
From eq. (21):
\[ r^2 \theta'' = \frac{L^2}{\gamma^2 m^2 r^3} \quad (24) \]

Using eqns. (23) and (24) in eq. (17):

\[ \frac{d^2}{d\theta^2} \left( \frac{1}{r} \right) + \frac{1}{r} = -\frac{8mr^3}{L^2} F(r) \quad (25) \]

and:

\[ F(r) = -\frac{L^2}{8mr^3} \left( \frac{d^2}{d\theta^2} \left( \frac{1}{r} \right) + \frac{1}{r} \right) \quad (26) \]

This is the relativistic boost expression. Q.E.D.

It is equivalent to:

\[ F(r) = \gamma m \left( \ddot{r} - r \dot{\theta}^2 \right) = -\frac{dV}{dr} \quad (27) \]

However, eq. (26) can be used to calculate the relativistic force for any position of the object.

Eq. (26) can be used to calculate \( F(r) \) using:
\[ F(r) = \frac{L^2}{mr^3} \left( 1 - \frac{v^2}{c^2} \right)^{1/2} \left( \frac{d^2}{d\theta^2} \left( \frac{1}{r} \right) + \frac{1}{r} \right) \]  

(28)

For which:

\[ v^2 = \left( \frac{ds}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \]  

(29)

\[ = \left( \frac{d\theta}{dt} \right)^2 \left( \left( \frac{ds}{d\theta} \right)^2 + r^2 \right) \]

\[ = \frac{L^2}{2mr^4 \left( \left( \frac{ds}{d\theta} \right)^2 + r^2 \right)} \]

\[ = \frac{L^2}{2mr^3 \left( 1 - \frac{v^2}{c^2} \right) \left( \left( \frac{ds}{d\theta} \right)^2 + r^2 \right)} \]  

(30)

so

\[ v^2 = \frac{L^2}{2mr^4} \left( 1 + \frac{L^2}{2mr^4} \left( r^2 + \left( \frac{ds}{d\theta} \right)^2 \right) \right)^{-1} \]

The observed are Newtonian forces at precursing canonical section, for example the precursing ellipse, where:
\[ r = \frac{d}{1 + e \cos(x\theta)} \]  
\text{At periapsis, } \min:\]
\[ d = (1 + e) \min \]  
\text{The relativistic force can be calculated by computer algebra using eqns (28), (30) and (31), and it can be plotted against the inverse square law:}
\[ F = -\frac{\text{m} \text{M}_B}{r^2} \]

for different \( x \).

From a comparison of eqs. (17) and (18) for planar orbits:
\[ \gamma m (\ddot{r} - r\dot{\theta}^2) \frac{\dot{r}}{r} = \gamma m \left( \frac{a}{c} + v \times \Omega \right) \]

so
\[ (\ddot{r} - r\dot{\theta}^2) \dot{r} = \frac{a}{c} + v \times \Omega \]

in this relativistic theory.