+15(2): The Orbital Equations in m Space

Consider the infinitesimal line element of m space:

$$ds^2 = c^2 d\tau^2 = c^2 m(r)\, dt^2 - \frac{dr^2}{m(r)} - r^2 d\phi^2 \quad -(1)$$

in plane polar coordinates (r, ϕ). The m space is defined as the most general spherically symmetric space. In general, m is a function of both r and t, but for simplicity it is defined as a function of r. Eq. (1) can be written as:

$$ds^2 = c^2 d\tau^2 = (c^2 - v_N^2)\, dt'^2 \quad -(2)$$

where

$$v_N^2 dt'^2 = \frac{dr^2}{m(r)} + r^2 d\phi^2 \quad -(3)$$

so

$$v_N^2 = \frac{1}{m(r)}\left(\frac{dr}{dt'}\right)^2 + r^2\left(\frac{d\phi}{dt'}\right)^2 \quad -(4)$$

in which:

$$\frac{dr}{dt'} = \frac{dr}{dt}\frac{dt}{dt'} = \frac{1}{m^{1/2}(r)}\frac{dr}{dt} \quad -(5)$$

and

$$\frac{d\phi}{dt'} = \frac{d\phi}{dt}\frac{dt}{dt'} = \frac{1}{m^{1/2}(r)}\frac{d\phi}{dt} \quad -(6)$$

So

$$v_N^2 = \frac{1}{m^2(r)}\left(\frac{dr}{dt}\right)^2 + \frac{r^2}{m(r)}\left(\frac{d\phi}{dt}\right)^2 \quad -(7)$$

is the square of the Newtonian velocity in m space. The transformation into m space is therefore defined by:

$$\left.\begin{array}{l} \dot{r} \rightarrow \frac{1}{m^{1/2}}\dot{r}, \quad \dot{\phi} \rightarrow \frac{1}{m^{1/2}}\dot{\phi}, \\ \ddot{r} \rightarrow \frac{1}{m^{1/2}}\ddot{r}, \quad \ddot{\phi} \rightarrow \frac{1}{m^{1/2}}\ddot{\phi} \end{array}\right\} \quad -(8)$$

) The relativistic kinematics of m space are now developed by modifying the results of Note 414(4) with the transformations (8). The velocity in m space is the relativistic velocity:

$$\underline{v} = \gamma \dot{\underline{r}} \quad -(9)$$

where

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(10)$$

and

$$v_N^2 = \frac{1}{m^2(r)} \dot{r}^2 + \frac{r^2}{m(r)} \dot{\phi}^2 \quad -(11)$$

The relativistic acceleration is:

$$\underline{a} = \ddot{\underline{r}} = \frac{d\underline{v}}{dt} = \frac{d}{dt}(\gamma \dot{\underline{r}}) \quad -(12)$$

$$= \frac{d\gamma}{dt} \dot{\underline{r}} + \gamma \ddot{\underline{r}}$$

Here,

$$\dot{\underline{r}} = \dot{r} \underline{e}_r + r \dot{\phi} \underline{e}_\phi \quad -(13)$$

$$= \dot{r} \underline{e}_r + r \frac{d \underline{e}_r}{dt}$$

in the line element

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(14)$$

and using eq. (8):

$$\dot{\underline{r}} = \frac{1}{m(r)^{1/2}} \left(\dot{r} \underline{e}_r + r \dot{\phi} \underline{e}_\phi \right) \quad -(15)$$

in the line element (1) of m theory. Similarly:

$$\ddot{\underline{r}} = \gamma \left(\left(\frac{\ddot{r}}{m^{1/2}(r)} - \frac{r \dot{\phi}^2}{m(r)} \right) \underline{e}_r + \left(\frac{r \ddot{\phi}}{m^{1/2}(r)} + \frac{2 \dot{r} \dot{\phi}}{m(r)} \right) \underline{e}_\phi \right) \quad -(16)$$

) in m space. So the orbital equation in m space is

$$\frac{1}{m(r)^{1/2}}\left(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\right)\frac{d\gamma}{dt}$$

$$+\gamma\left(\left(\frac{\ddot{r}}{m(r)^{1/2}} - \frac{r\dot{\phi}^2}{m(r)}\right)\underline{e}_r + \left(\frac{r\ddot{\phi}}{m(r)^{1/2}} + \frac{2\dot{r}\dot{\phi}}{m(r)}\right)\underline{e}_\phi\right)$$

$$= -\frac{mG}{r^2}\,\underline{e}_r \quad -(17)$$

The orbit in m space is therefore given by simultaneous solution of:

$$\frac{1}{m(r)^{1/2}}\frac{d\gamma}{dt}\dot{r} + \gamma\left(\frac{\ddot{r}}{m(r)^{1/2}} - \frac{r\dot{\phi}^2}{m(r)}\right) = -\frac{mG}{r^2} \quad -(18)$$

and

$$\frac{1}{m(r)^{1/2}}r\dot{\phi}\frac{d\gamma}{dt} + \gamma\left(\frac{r\ddot{\phi}}{m(r)^{1/2}} + \frac{2\dot{r}\dot{\phi}}{m(r)}\right) = 0 \quad -(19)$$

Eq. (18) is the Leibniz equation in m space and Eq. (19) is the conservation of angular momentum in m space.

Eq. (19) is

$$r\dot{\phi}\frac{d\gamma}{dt} + \gamma\left(r\ddot{\phi} + \frac{2\dot{r}\dot{\phi}}{m^{1/2}(r)}\right) = 0 \quad -(20)$$

The relativistic angular momentum of m theory is:

$$\boxed{L = \frac{\gamma}{m(r)^{1/2}}\,mr^2\dot{\phi}} \quad -(21)$$

t) This is a constant of motion, so:

$$\frac{dL}{dt} = 0 \quad -(22)$$

From eqs. (21) and (22):

$$\frac{dL}{dt} = \frac{mr^2\dot{\phi}}{n(r)^{1/2}}\frac{d\gamma}{dt} + \gamma m\frac{d}{dt}\left(\frac{r^2\dot{\phi}}{n(r)^{1/2}}\right) = 0 \quad -(23)$$

Since $n(r)$ does not depend on time, eq. (23) is:

$$\frac{mr^2\dot{\phi}}{n(r)^{1/2}}\frac{d\gamma}{dt} + \frac{\gamma m}{n(r)^{1/2}}\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0$$

i.e.

$$r^2\dot{\phi}\frac{d\gamma}{dt} + \gamma\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0 \quad -(24)$$

and

$$r\dot{\phi}\frac{d\gamma}{dt} + \gamma\left(2\frac{\dot{r}}{n(r)^{1/2}}\dot{\phi} + r\ddot{\phi}\right) = 0$$

which is eq. (20), Q.E.D. In deriving eq. (24) we have used:

$$\dot{r} \rightarrow \frac{\dot{r}}{n(r)^{1/2}} \quad -(25)$$

Note carefully that

$$\dot{\phi} \rightarrow \frac{\dot{\phi}}{n(r)^{1/2}} \quad -(26)$$

has already been used to derive eq. (23).

The derivation of the n space relativistic angular momentum of eq. (21) can be checked using the Lagrangian method:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \quad -(27)$$

$$= -mc^2\left(1 - \frac{1}{c^2 n(r)^{1/2}}\left(\frac{\dot{r}^2}{n(r)^{1/2}} + r^2\dot{\phi}^2\right)\right)^{1/2} + \frac{mMG}{r}$$

and the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(28)$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(29)$$

From eqs. (27) and (29):

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\gamma}{n(r)^{1/2}} m r^2 \dot{\phi} \quad -(30)$$

which is eq. (21), Q.E.D. From eq. (29)

$$\frac{dL}{dt} = 0 \quad -(31)$$

Q.E.D.

The orbit of m theory is obtained by solving eqs. (18) and (19) simultaneously for a given $n(r)$.