414(6): Orbital Equations in Plane Polar Coordinates

The relativistic orbit is given by simultaneous solution of the relativistic force equation:

$$F = m\gamma^3 \dot{v} = -\frac{mMG}{r^2} \quad -(1)$$

and conservation of relativistic angular momentum:

$$\frac{dL}{dt} = 0 \quad -(2)$$

where

$$L = \gamma m r^2 \dot{\phi} \quad -(3)$$

In these equations the frame has not yet been rotated, so these are the baseline computations.

In plane polar coordinates:

$$\gamma = \left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)\right)^{-1/2} \quad -(4)$$

Equations (2) and (3) give:

$$r\dot{\phi}\frac{d\gamma}{dt} + \gamma\left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right) = 0 \quad -(5)$$

The Hamiltonian method of Note 414(5) gives:

$$\frac{d\gamma}{dt} = -\frac{\dot{r}}{c^2}\frac{MG}{r^2} \quad -(6)$$

so eq. (5) is:

$$\gamma\left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right) = \frac{r\dot{r}\dot{\phi}}{c^2}\frac{MG}{r^2} \quad -(7)$$

The orbit is therefore given by simultaneous solution of eqns. (1) and (7), with γ defined

) by eq. (4).

The non relativistic limit of eqs. (1) and (7) is:

$$F = m\dot{v} = -\frac{mMG}{r^2} \quad -(8)$$

and

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = 0 \quad -(9)$$

As in note 44(4):

$$\dot{v} = \ddot{r} - r\dot{\phi}^2 \quad -(10)$$

Eqs. (8) to (10) give the conic section, for example an ellipse, parabola, hyperbola and circle. Therefore the relativistic orbit is found from simultaneous solution of:

$$F = m\left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)\right)^{-3/2}\left(\ddot{r} - r\dot{\phi}^2\right) = -\frac{mMG}{r^2} \quad -(10)$$

and

$$\left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)\right)^{-1/2}\left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right) = \frac{r\dot{r}\dot{\phi}}{c^2}\frac{MG}{r^2} \quad -(11)$$

These two equations can be reduced to one by dividing eq. (1) by eq. (7) to give:

$$\frac{\gamma^2\dot{v}}{2\dot{r}\dot{\phi} + r\ddot{\phi}} = -\frac{c^2}{r\dot{r}\dot{\phi}} \quad -(12)$$

i.e.

$$\frac{\gamma^2}{c^2}\dot{v}\, r\dot{r}\dot{\phi} + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right) = 0 \quad -(13)$$

or

$$\frac{\gamma^2}{c^2}\left(\ddot{r} - r\dot{\phi}^2\right) r\dot{r}\dot{\phi} + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right) = 0 \quad -(14)$$

3) It is seen that eq. (14) is a small relativistic correction to the classical

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = 0 \quad -(15)$$

Eq. (14) is one equation in two unknowns r and ϕ, so another equation is needed. This is given by the hamiltonian

$$H = \gamma m c^2 - \frac{mMG}{r} \quad -(16)$$

and

$$\frac{dH}{dt} = 0 \quad -(17)$$

As in note 414(4), Eq. (6) gives:

$$c^2 \frac{d\gamma}{dt} = -\dot{r}\frac{MG}{r^2} \quad -(18)$$

As in Note 414(4):

$$\frac{d\gamma}{dt} = \gamma^3 \frac{v}{c^2}\frac{dv}{dt} \quad -(19)$$

$$= \frac{\gamma^3 \dot{r}}{c^2}\left(\ddot{r} - r\dot{\phi}^2\right)$$

so

$$\gamma^3\left(\ddot{r} - r\dot{\phi}^2\right) = -\frac{MG}{r^2} \quad -(20)$$

which is the force equation:

$$F = \gamma^3 m\dot{v} = -\frac{mMG}{r^2} \quad -(21)$$

However, using eq. (20) in eq. (14) gives eq. (7), so eqs. (14) and (20) are not independent.

The truly independent equations are:

$$H = \gamma m c^2 - \frac{m M G}{r} \quad -(22)$$

$$F = m\gamma^3 (\ddot{r} - r\dot{\phi})^r = -\frac{m M G}{r^2} \quad -(23)$$

and

$$\gamma(2\dot{r}\dot{\phi} + r\ddot{\phi}) = \frac{r\dot{r}\dot{\phi}}{c^2}\frac{MG}{r^2} \quad -(24)$$

Eq. (24) has been derived using:

$$\frac{dH}{dt} = 0 \quad -(25)$$

as in Note 414(5).

The two Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(26)$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(27)$$

are truly independent and give:

$$\boxed{\gamma^3(\ddot{r} - r\dot{\phi}) = -\frac{MG}{r^2} \quad -(28)}$$

and

$$\boxed{r\dot{\phi}\frac{d\gamma}{dt} + \gamma(2\dot{r}\dot{\phi} + r\ddot{\phi}) = 0 \quad -(29)}$$

respectively.

Eqs. (28) and (29) can be solved for r and ϕ, giving the orbit in the unrotated frame.

In these equations:

$$\gamma^3 = \left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)\right)^{-3/2} \quad -(30)$$

and

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)\right)^{-1/2} \quad -(31)$$

Let

$$f = 1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right), \quad -(32)$$

so:

$$\gamma = f^{-1/2} \quad -(33)$$

and

$$\frac{d\gamma}{df} = -\frac{1}{2} f^{-3/2} = -\frac{\gamma^3}{2}. \quad -(34)$$

Note that

$$\frac{d\gamma}{dt} = \frac{d\gamma}{df}\frac{df}{dt} = -\frac{\gamma^3}{2}\frac{df}{dt} \quad -(35)$$

where:

$$\frac{df}{dt} = -\frac{1}{c^2}\frac{d}{dt}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \quad -(36)$$

in which:

$$\frac{d}{dt}\left(\dot{r}^2\right) = \frac{d}{d\dot{r}}\dot{r}^2\frac{d\dot{r}}{dt} = 2\dot{r}\ddot{r} \quad -(37)$$

and

$$\frac{d}{dt}\left(r^2\dot{\phi}^2\right) = \dot{\phi}^2\frac{d}{dt}r^2 + r^2\frac{d}{dt}\dot{\phi}^2 \quad -(38)$$

with

$$\frac{d}{dt}r^2 = \frac{dr^2}{dr}\frac{dr}{dt} = 2r\dot{r} \quad -(39)$$

and

$$\frac{d}{dt}\dot{\phi}^2 = \frac{d}{d\dot{\phi}}\dot{\phi}^2\frac{d\dot{\phi}}{dt} = 2\dot{\phi}\ddot{\phi} \quad -(40)$$

so

$$\frac{df}{dt} = -\frac{2}{c^2}\left(\dot{r}\ddot{r} + r\dot{r}\dot{\phi}^2 + \dot{\phi}\ddot{\phi}r^2\right) -(41)$$

and

$$\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2}\left(\dot{r}\ddot{r} + r\dot{r}\dot{\phi}^2 + r^2\dot{\phi}\ddot{\phi}\right)$$

Therefore eq. (29) is:

$$\frac{\dot\phi}{c^2}\gamma^3\left(\dot r\ddot r + r\dot r\dot\phi^2 + r^2\dot\phi\ddot\phi\right) + \gamma\left(2\dot r\dot\phi + r\ddot\phi\right) = 0 \qquad -(43)$$

i.e.

$$\boxed{2\dot r\dot\phi + r\ddot\phi + \frac{r\dot\phi\gamma^2}{c^2}\left(\dot r\ddot r + r\dot r\dot\phi^2 + r^2\dot\phi\ddot\phi\right) = 0 \qquad -(44)}$$

which must be solved simultaneously with:

$$\boxed{\gamma^3\left(\ddot r - r\dot\phi\right) = -\frac{MG}{r^2} \qquad -(45)}$$

These are two truly independent equations in r and ϕ, which must be solved simultaneously to give the orbit.

In the classical limit they reduce to:

$$\ddot r - r\dot\phi = -\frac{MG}{r^2} \qquad -(46)$$

and

$$2\dot r\dot\phi + r\ddot\phi = 0 \qquad -(47)$$

which give a static ellipse. Eqs. (45) and (44) give relativistic effects in the non rotating frame.

Eq. (44) is the conservation of relativistic angular momentum:

$$\frac{dL}{dt} = \frac{d}{dt}\left(\gamma m r^2\dot\phi\right) = 0 \qquad -(48)$$

and eq. (45) is the relativistic Leibniz equation.