411(1): Development of the Universal Law of Precessions for Decreasing Orbits.

Following Note 409(5) the basic equations of dynamics are developed in plane polar coordinates. This method gives a basic insight into the meaning of the angular velocity of the universal law of precessions. The key feature of the plane polar coordinate system is that the frame of reference is rotating. The unit vectors of the system are given by:

$$\underline{e}_r = \underline{i}\cos\phi + \underline{j}\sin\phi \quad -(1)$$

$$\underline{e}_\phi = -\underline{i}\sin\phi + \underline{j}\cos\phi \quad -(2)$$

in the notation of Note 409(5).

The dynamics of the coordinate system are defined as

$$\frac{d\underline{e}_r}{d\phi} = \underline{e}_\phi \quad -(3)$$

$$\frac{d\underline{e}_\phi}{d\phi} = -\underline{e}_r \quad -(4)$$

Eqs. (3) and (4) follow directly from Eqs. (1) and (2). The position vector of the plane polar system is:

$$\underline{r} = r\,\underline{e}_r \quad -(5)$$

$$= X\underline{i} + Y\underline{j}$$

The velocity is defined by:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\,\underline{e}_r) \quad -(6)$$

Using the Leibniz Theorem:

$$\frac{d}{dt}(r\,\underline{e}_r) = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt} \quad -(7)$$

because $\underline{e}_r$ depends on time. In the Cartesian system:

$$\underline{v} = \dot{X}\underline{i} + \dot{Y}\underline{j} \quad -(8)$$

2) and $\frac{d\underline{i}}{dt} = \frac{d\underline{j}}{dt} = 0$ — (9)

The axes of the Cartesian system are fixed and are not time dependent.

From eq. (3):

$$\frac{d\underline{e}_r}{dt} = \frac{d\phi}{dt}\underline{e}_\phi \quad - (10)$$

$$= \omega\,\underline{e}_\phi$$

where

$$\omega = \frac{d\phi}{dt} \quad - (11)$$

is the angular velocity. Similarly, from eq. (4):

$$\frac{d\underline{e}_\phi}{dt} = -\omega\,\underline{e}_r \quad - (12)$$

From eqs. (7) and (10):

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dr}{dt}\underline{e}_r + \omega r\,\underline{e}_\phi \quad - (13)$$

$$= \underline{v}_r + \underline{v}_\phi$$

where

$$\underline{v}_r = \frac{dr}{dt}\underline{e}_r \quad - (14)$$

$$\underline{v}_\phi = \omega r\,\underline{e}_\phi \quad - (15)$$

So

$$|\underline{v}_r| = \frac{dr}{dt} = v_r \quad - (16)$$

$$|\underline{v}_\phi| = \omega r = v_\phi \quad - (17)$$

Using:

$$\underline{e}_\phi = \underline{k} \times \underline{e}_r \quad - (18)$$

$$\underline{\omega} = \omega\,\underline{k} \quad - (19)$$

$$\underline{r} = r\,\underline{e}_r \quad - (20)$$

3) it is found that:

$$\dot{\underline{r}} = \underline{v} = \frac{dr}{dt}\,\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(21)$$

$$= \dot{X}\,\underline{i} + \dot{Y}\,\underline{j}$$

It can be shown that eq. (21) is true for any vector $\underline{V}$ of classical dynamics:

$$\dot{\underline{V}} = \frac{dV}{dt}\,\underline{e}_r + \underline{\omega} \times \underline{V} \quad -(22)$$

$$= \dot{V}_x\,\underline{i} + \dot{V}_y\,\underline{j}$$

Notably, the acceleration is:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{dv}{dt}\,\underline{e}_r + \underline{\omega} \times \underline{v} \quad -(23)$$

Denoting the angular momentum by $\underline{J}$, the torque $\underline{Tq}$ is:

$$\underline{Tq} = \frac{d\underline{J}}{dt} = \frac{dJ}{dt}\,\underline{e}_r + \underline{\omega} \times \underline{J} \quad -(24)$$

The angular acceleration is:

$$\frac{d\underline{\omega}}{dt} = \frac{d\omega}{dt}\,\underline{e}_r + \underline{\omega} \times \underline{\omega} \quad -(25)$$

$$= \frac{d\omega}{dt}\,\underline{e}_r$$

Eq. (24) is important for the dynamics of gyroscopes and it will be shown in UFT 311 that the angular acceleration is of key importance for explaining the shrinking of orbits in a binary pulsar.

From eq. (21) and eq. (23):

4)

$$\underline{a} = \frac{dv}{dt}\underline{e}_r + \underline{\omega} \times \left(\frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r} \right) - (26)$$

$$= \frac{dv}{dt}\underline{e}_r + \underline{\omega} \times \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$= \dot{v}_X \underline{i} + \dot{v}_Y \underline{k}$$

Eq. (26) defines the inertial, Coriolis and centripetal accelerations. Eq. (26) can be expressed as:

$$\underline{a} = (\ddot{r} - r\dot{\phi}^2)\underline{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\underline{e}_\phi - (27)$$

$$= \ddot{X}\underline{i} + \ddot{Y}\underline{j}$$

The above well known equations are the result of intrinsic frame rotation of the plane polar coordinate system and are defined at the classical level.

The universal law of precession is the result of considering the ECE covariant infinitesimal line element in plane polar coordinates:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - (28)$$

$$= c^2 d\tau^2 = (c^2 - v_N^2)\, dt^2$$

where v_N^2 is the Newtonian velocity:

$$v_N^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 - (29)$$

$$= \dot{X}^2 + \dot{Y}^2$$

The frame is rotated using:

$$d\phi \rightarrow d\phi + \omega\, dt - (30)$$

producing the result:

$$ds^2 = c^2 d\tau^2 = c^2\left(1 - 3\frac{v_N^2}{c^2}\right)dt^2 - (31)$$

5) As in UFT 310 this produces the precession

$$\Delta \phi = \frac{2\pi}{c^2}\left(v_N^2 + 3\omega^2 r^2\right) \quad -(32)$$

and for

$$\phi \rightarrow \phi - \omega t \quad -(33)$$

$$\Delta \phi = \frac{2\pi}{c^2}\left(v_N^2 - \omega^2 r^2\right) \quad -(34)$$

The additional rotation (30) means that (10) is changed to:

$$\frac{d\underline{e}_r}{dt} = \left(\frac{d\phi}{dt} + \frac{d}{dt}(\omega t)\right)\underline{e}_\phi \quad -(35)$$

$$= \left(\frac{d\phi}{dt} + \omega + t\frac{d\omega}{dt}\right)\underline{e}_\phi$$

$$= \left(2\omega + t\frac{d\omega}{dt}\right)\underline{e}_\phi$$

and

$$\frac{d\underline{e}_\phi}{dt} = -\left(2\omega + t\frac{d\omega}{dt}\right)\underline{e}_r \quad -(36)$$

This means that the additional frame rotation (30) produces many new dynamical results on the classical level, so eqs. (21) to (27) are all affected. It has been shown that precession is an effect of the vacuum, so these are all effects of the vacuum.

Finally note that eq. (30) produces:

$$d\phi \rightarrow d\phi + d(\omega t) \quad -(37)$$

so

$$\boxed{d\phi' = d\phi + \omega\, dt + t\, d\omega} \quad -(38)$$

so a non-zero angular acceleration produces new dynamical and orbital effects to be discussed in the next notes.