404(6) : Spin Connection of the Precession due to Rotation

In the apsidal method the precession due to rotation is defined by

$$\Delta \phi = \frac{r^2}{2} \left(\frac{\omega}{r} - \frac{\partial \omega}{\partial r} \right) \quad -(1)$$

where ω is the magnitude of the spin connection defined by

$$\underline{g} = - \underline{\nabla} \phi + \underline{\omega} \phi \quad -(2)$$

and where ϕ is the gravitational potential:

$$\phi = - \frac{MG}{r} \quad -(3)$$

where M is the mass of the Earth. In the Larmor method of 4FT345

$$\Delta \phi = \frac{1}{2} | \underline{\Omega} | t \quad -(4)$$

where $\underline{\Omega}$ is the Earth's gravitomagnetic field and t is the number of seconds in a year:

$$t = 24 \times 3600 \times 365.25 \quad -(5)$$

The units of $\Delta \phi$ in eq. (1) are radians per year, and the units of $\Delta \phi$ in eq. (4) are also radians per year.

The gravitomagnetic field is:

$$\underline{\Omega} = \frac{G}{2c^2 r^3} \left(3 \underline{L} \cdot \frac{\underline{r}\,\underline{r}}{r^2} - \underline{L} \right) \quad -(6)$$

in units of radians per second. The modulus of $\underline{\Omega}$ is

$$\Omega = \frac{G}{2c^2 r^3} \left(\left(3 \underline{L} \cdot \frac{\underline{r}\,\underline{r}}{r^2} - \underline{L} \right) \cdot \left(3 \underline{L} \cdot \frac{\underline{r}\,\underline{r}}{r^2} - \underline{L} \right) \right)^{1/2} \quad -(7)$$

Equating eqns. (1) and (4):

2)

$$\boxed{\frac{r^2}{2}\left(\frac{\omega}{r}-\frac{\partial\omega}{\partial r}\right)=\frac{1}{2}\Omega t} \quad -(8)$$

In UFT345 the gravitomagnetic field is defined as:

$$\underline{\Omega}=\frac{2}{5}\frac{MGR^2}{c^2r^3}\left(\underline{\omega}_E-3\underline{n}\left(\underline{\omega}_E\cdot\underline{n}\right)\right) \quad -(9)$$

where $\underline{\omega}_E$ is the angular velocity vector of the Earth and

$$\underline{n}:=\frac{\underline{r}}{r} \quad -(10)$$

Here

$$\underline{\omega}_E=\omega_E\underline{k} \quad -(11)$$

because the Earth spins around the $\underline{k}$ axis $\perp$ equator. Gravity Probe B orbited in a plane perpendicular to the equator so

$$\underline{r}=Y\underline{j}+Z\underline{k} \quad -(12)$$

and

$$\underline{\omega}_E-3\underline{n}\left(\underline{\omega}_E\cdot\underline{n}\right)=\omega_E\left(\left(1-\frac{3Z^2}{r^2}\right)\underline{k}-\frac{3YZ}{r^2}\underline{j}\right) \quad -(13)$$

therefore:

$$\underline{\Omega}=\frac{2}{5}\frac{MGR^2\omega_E}{c^2r^3}\left(\left(1-\frac{3Z^2}{r^2}\right)\underline{k}-\frac{3YZ}{r^2}\underline{j}\right) \quad -(14)$$

as in UF345, Eq. (6).

The Gravity Probe B spacecraft carried precision gyroscopes which are currents of mass and can be regarded as gravitomagnetic dipole moments $\underline{m}$. This generated the torque:

$$\underline{Tq}=\underline{m}\times\underline{\Omega} \quad -(15)$$

and the Larmor precession frequency:

$$\Omega=\frac{1}{2}|\underline{\Omega}| \quad -(16)$$

The relevant quantities used in UFT345 were defined as follows:

$$\underline{M} = 5.98 \times 10^{24}\ kg$$
$$R = 6.37 \times 10^{6}\ m$$
$$r = 7.02 \times 10^{6}\ m$$
$$c = 2.998 \times 10^{8}\ m s^{-1}$$
$$G = 6.67 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$$
$$\omega_E = 7.292 \times 10^{-5}\ rad\ s^{-1}$$

At the equator:

$$\underline{\omega}_E \cdot \underline{n} = 0 \quad -(17)$$

and at the equator:

$$\Omega = 1.52 \times 10^{-14}\ rad\ s^{-1} \quad -(18)$$

compared with the experimental value (UFT117) of

$$\Omega(\exp) = 1.26 \times 10^{-14}\ rad\ s^{-1} \quad -(19)$$

More generally

$$\underline{\omega}_E \cdot \underline{n} = \frac{Z}{r}\,\omega_E\,\underline{k} \quad -(20)$$

and

$$3\underline{n}\left(\underline{\omega}_E \cdot \underline{n}\right) = \frac{3Z\omega_E}{r^2}\left(Y\underline{j} + Z\underline{k}\right) \quad -(21)$$

It follows that:

$$\underline{\omega} - 3\underline{n}\left(\underline{\omega}\cdot\underline{n}\right) = \omega\underline{k} - 3\omega\frac{Z}{r}\left(\frac{Y}{r}\underline{j} + \frac{Z}{r}\underline{k}\right) \quad -(22)$$

where

$$\sin\theta = \frac{Z}{r}, \quad \cos\theta = \frac{Y}{r} \quad -(23)$$

so

$$\Omega = \frac{MGR^2\omega_E}{5c^2r^3}\left\|\left(1 - 3\sin^2\theta\right)\underline{k} - 3\sin\theta\cos\theta\,\underline{j}\right\|$$

$$= \frac{MGR^2\omega_E}{5c^2r^3}\left(\left(1-3\sin^2\theta\right)^2 + 9\sin^2\theta\cos^2\theta\right)^{1/2} \quad -(24)$$

In Section 3 of UFT345 it was shown that

4) The experimental claim from Gravity Probe B is

$$\Omega_{exp} = 0.0372 \text{ arc seconds per year} \quad -(17)$$

whereas:

$$\Omega = \frac{2}{5}\frac{MGR^2}{c^2 r^3} \quad -(18)$$

gives:

$$\Omega = 0.099 \text{ arc seconds per year}$$
$$= 1.52 \times 10^{-14} \text{ rad s}^{-1}$$

The Ω from eq. (24) can be averaged to give:

$$\langle \Omega \rangle = \Omega_{exp} \quad -(19)$$

In these calculations we have used:

$$\text{One year} = 3.156 \times 10^{7} \text{ seconds} \quad -(20)$$
$$\text{One radian} = 2.06265 \times 10^{5} \text{ arc seconds} \quad -(21)$$

The experimental claim is:

$$\Delta\phi = \frac{0.037}{2.06265} \times 10^{-5} \text{ radians per year}$$
$$= 1.794 \times 10^{-7} \text{ radians per year} \quad -(22)$$

So

$$\Delta\phi = \frac{r^2}{2}\left(\frac{\omega}{r} - \frac{\partial \omega}{\partial r}\right) = 1.794 \times 10^{-7} \text{ rad per year}$$
$$= \frac{1}{3}\left(4\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} - \frac{1}{r}\frac{\partial}{\partial r}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle\right) \quad -(23)$$

using eq. (42) of UFT 403. In the limit of an exactly circular orbit:

$$\Delta\phi = 0 \quad -(24)$$

so eq. (?) of UFT 403:

$$\frac{\partial}{\partial r}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle \sim \frac{4}{a}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle \quad -(25)$$

and

$$\frac{1}{3}\left(4\frac{\langle \delta \underline{r}\cdot\delta \underline{r}\rangle}{r^2} - \frac{4}{ar}\langle \delta \underline{r}\cdot\delta \underline{r}\rangle\right)$$

$$= 1.794\times 10^{-7} \quad -(26)$$

i.e

$$\boxed{\frac{4}{3}\frac{\langle \delta \underline{r}\cdot\delta \underline{r}\rangle}{r}\left(\frac{1}{r}-\frac{1}{a}\right) = 1.794\times 10^{-7}}$$

$$-(27)$$

For gravity probe B:

$$a = 7.0274\times 10^{6}\,m \quad -(28)$$

$$b = 7.02739\times 10^{6}\,m \quad -(29)$$

$$\epsilon = 0.0014 \quad -(30)$$

So, for example, if r is chosen to be the perihelion, the distance of closest approach of GPB to the centre of the Earth, then:

$$r = \frac{\alpha}{1+\epsilon} \sim \frac{a}{1+\epsilon} \quad -(31)$$

and

$$\frac{4}{3}\frac{\langle \delta \underline{r}\cdot\delta \underline{r}\rangle}{a}(1+\epsilon)\left(\frac{\epsilon}{a}\right) = 1.794\times 10^{-7} \quad -(32)$$

and

$$\frac{\langle \delta \underline{r}\cdot\delta \underline{r}\rangle}{a^2} = \frac{3}{4\epsilon(1+\epsilon)}\times 1.794\times 10^{-7} \quad -(33)$$

$$\sim 10^{-5}$$