...t(4): Final Version of Note 404(2).

The spin connections are given by eqs. (26) and (29) of Note ...(2):

$$\omega = \mp \frac{r v}{2Mc^2} \frac{df(r)}{dr} \quad -(1)$$

...ere

$$v = \frac{dr}{dt} \quad -(2)$$

$$f(r) = \frac{1}{r^3}\left(L^2 r^2 - (\underline{L} \cdot \underline{r})^2\right)^{1/2}. \quad -(3)$$

To an excellent approximation, Newtonian dynamics can be ...ed to calculate v. The hamiltonian is:

$$H = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L^2}{mr^2} + U(r) \quad -(4)$$

...ere

$$U(r) = -\frac{mMG}{r} \quad -(5)$$

...ere m is the mass of the satellite, M is the mass of the earth, ...d r is the distance between the centre of the earth and the satellite. Gravity Probe B orbited with:

$$d = 7.0274 \times 10^5 \text{ m} \quad -(6)$$

$$\epsilon = 0.0014 \quad -(7)$$

$$\text{Perigee} = 6.41 \times 10^5 \text{ m} \quad -(8)$$

$$\text{Apogee} = 6.45 \times 10^5 \text{ m} \quad -(9)$$

$$T = 97.65 \text{ minutes} \quad -(10)$$

The mean distance from the centre of the earth to the satellite was

$$r = \langle r \rangle = 7.02 \times 10^6 \text{ m}, \quad -(11)$$

so

$$\frac{dr}{dt} = \left(\frac{2}{m}(H - U) - \frac{L^2}{m^2 r^2}\right)^{1/2} \quad -(12)$$

For an elliptical orbit:

$$H < 0 \quad -(13)$$

and

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{mMG}{2|H|} \quad -(14)$$

Here a is the semi major axis:

$$a = 3.275 \times 10^5 m \quad -(15)$$

The mean radius of the earth is:

$$r_E = 6.37 \times 10^6 m \quad -(16)$$

so

$$r - r_E = 6.50 \times 10^5 m \quad -(17)$$

There is a considerable discrepancy in the literature between eqs (17) and eqs (8) and (9), which reduces confidence in claims of the standard model to very high precision.

From eq. (14):

$$|H| = \frac{mMG}{2a} \quad -(18)$$

and

$$H = -\frac{mMG}{2a} \quad -(19)$$

so:

$$\frac{dr}{dt} = \left(2MG\left(\frac{1}{r} - \frac{1}{2a}\right) - \frac{L^2}{m^2 r^2}\right)^{1/2}$$

$$= \left(MG\left(\frac{2}{r} - \frac{1}{a}\right) - \frac{L^2}{m^2 r^2}\right)^{1/2} \quad -(20)$$

The angular momentum is:

$$L = \frac{2}{5} M r_E^2 \omega_E \quad -(21)$$

3) where:

$$\omega_E = 7.292 \times 10^{-5} \text{ rad } s^{-1} \quad -(22)$$

To an excellent approximation:

$$r = a = 7.02 \times 10^{6} \text{ m} \quad -(23)$$

and

$$-(24)$$

$$L = mr^2 \frac{d\phi}{dt} \quad -(25)$$
$$= \text{constant}$$

So:

$$\frac{dr}{dt} = \left(\frac{mG}{r} - r^2 \left(\frac{d\phi}{dt} \right)^2 \right)^{1/2} \quad -(26)$$

In the approximation (23) the orbital velocity is:

$$v^2 = \left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 = mG \left(\frac{2}{r} - \frac{1}{a} \right) \quad -(27)$$
$$\sim \frac{mG}{r}$$

The angular momentum is given by:

$$b = \frac{\alpha}{(1-\epsilon^2)^{1/2}} = \frac{L}{(2m|H|)^{1/2}} \quad -(28)$$

where b is the semi minor axis. So:

$$L^2 = 2m|H|b^2 = m^2 MG \frac{b^2}{a} \quad -(29)$$

So:

$$\boxed{\frac{dr}{dt} = \left(2mG \left(\frac{1}{r} - \frac{1}{2a} \right) - \frac{MG}{r^2} \frac{b^2}{a} \right)^{1/2}}$$

4) If

$$a = b \quad - (31)$$

then

$$\frac{dr}{dt} = 0 \quad - (32)$$

for a circular orbit. Therefore the apsidal precession depends on deviations from circularity.

In eq. (30):

$$a = \frac{d}{1-\epsilon^2}, \quad b = \frac{d}{(1-\epsilon^2)^{1/2}} \quad - (33)$$

and at the perihelion:

$$r = \frac{d}{1+\epsilon} \quad - (34)$$

so dr / dt can be calculated given d and ϵ of the orbit.

In the rough approximation:

$$r \sim d \sim a \quad - (35)$$

eq. (30) reduces to:

$$\left(\frac{dr}{dt}\right)^2 \sim \frac{MG}{d}\left(1 - \frac{b^2}{d a}\right) \quad - (36)$$

The orbital parameters given by wikipedia and used in eqs. (6) to (10) relate to distances above the surface of the earth. However r relates to the distance from the centre of the earth, so we use:

$$d \rightarrow r_E + d, \quad b \rightarrow r_E + b, \quad a \rightarrow r_E + a \quad - (37)$$

where r_E is the earth's radius. This gives

$$\frac{dr}{dt} = 7.51 \times 10^2 \ m\, s^{-1} \quad - (38)$$

5) This value compares with:

$$V_E = 4.60 \times 10^2 \ ms^{-1} \quad -(39)$$

for the velocity of a point on the Earth's surface with respect to the centre of the Earth. So the initial guess (39) in Note 404(2) was quite accurate.

Finally using eq. (51) of Note 404(2):

$$\Delta\phi \sim \frac{1}{5} \frac{\omega_E V r}{c^2} \quad -(40)$$

it is found from eq. (38) that:

$$\Delta\phi = 0.86 \times 10^{-13} \text{ rad per year} \quad -(41)$$

compared with the experimental claim for Gravity Probe B of

$$\Delta\phi = 1.02 \times 10^{-12} \text{ rad per year} \quad -(42)$$