403(4): Final Version of Note 403(3)

Consider from Note 403(3):

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r_2}\right)+\frac{1}{r_2} = \frac{\omega_r}{1+\epsilon\cos\phi} \quad -(1)$$

In planetary systems, the precession is very small, so the spin connection ω_r is very small. The solution:

$$\frac{1}{r_2} \sim \frac{\omega_r}{1+\epsilon\cos\phi} \quad -(2)$$

implies:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r_2}\right) \sim 0 \quad -(3)$$

From Note 403(3), eq. (3) implies:

$$\frac{1}{r_2} = -\frac{2\omega_r}{\epsilon}\frac{\cos\phi}{\sin^2\phi} \quad -(4)$$

$$= \frac{\omega_r}{1+\epsilon\cos\phi}$$

Eq. (4) is approximately true if:

$$\omega_r \longrightarrow 0 \quad -(5)$$

because it reduces to:

$$0 \sim 0 \quad -(6)$$

So to an excellent approximation:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos\phi\right)+\frac{\omega_r}{1+\epsilon\cos\phi} \quad -(7)$$

is the particular integral. The complementary function is the solution of:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r_2}\right) + \frac{1}{r_2} = 0 \quad -(8)$$

i.e.

$$\frac{1}{r_2} = C_1 \cos\phi + C_2 \sin\phi \quad -(9)$$

where C_1 and C_2 are arbitrary constants. So the general solution of:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}\left(1 + \omega_r r\right) \quad -(10)$$

is:

$$\boxed{\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\phi\right) + \frac{\omega_r}{1+\epsilon\cos\phi} + C_1\cos\phi + C_2\sin\phi} \quad -(11)$$

In this solution ϵ is kept constant.

Checks

If:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\phi\right) \quad -(12)$$

then

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha} \quad -(13)$$

which is the conic section term on the right hand side of Eq. (11)

If

$$\frac{1}{r} = C_1\cos\phi + C_2\sin\phi \quad -(14)$$

then

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = 0 \quad -(15)$$

3)

3) Recall that the successive approximation method is to first consider the trial function:

$$\frac{1}{r} = \frac{1}{r_1} = \frac{1}{\alpha}\left(1+\epsilon\cos\phi\right) \quad -(16)$$

to see the solution of eq. (10). When used in the right hand side of eq. (10), eq. (16) produces:

$$\frac{1}{\alpha}\left(1+r_1\omega_r\right) = \frac{1}{\alpha} + \frac{\omega_r}{1+\epsilon\cos\phi} \quad -(17)$$

The second trial function is:

$$\frac{1}{r} = \frac{1}{r_2} + \frac{1}{r_1} \quad -(18)$$

with

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r_1}\right) + \frac{1}{r_1} = \frac{1}{\alpha} \quad -(19)$$

and

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r_2}\right) + \frac{1}{r_2} = \frac{\omega_r}{1+\epsilon\cos\phi} \quad -(20)$$

so

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}\left(1+\omega_r r_1\right) \quad -(21)$$