403(2): Precession from the ECE2 Force Equation of Orbits

Consider the ECE2 force equation of orbits:

$$\underline{F} = m\ddot{\underline{r}} = -\underline{\nabla}\phi + \underline{\omega}\phi \quad -(1)$$

where

$$\phi = -\frac{mMG}{r} \quad -(2)$$

is the gravitational potential and $\underline{\omega}$ is the spin connection. In general, in plane polar coordinates (r, ϕ):

$$\underline{\omega} = \omega_r \underline{e}_r + \omega_\phi \underline{e}_\phi \quad -(3)$$

so eq. (1) becomes:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{MG}{r^2} - \frac{MG}{r}\omega_r \quad -(4)$$

and

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = -\frac{MG}{r}\omega_\phi \quad -(5)$$

From eq. (4):

$$F = -\frac{mMG}{r^2}\left(1 + \omega_r r\right) \quad -(6)$$

The vacuum produces the force:

$$F(\text{vac}) = -\frac{mG\omega_r}{r} = -\frac{\partial U(r)}{\partial r} \quad -(7)$$

so

$$U(\text{vac}) = mG\int \frac{\omega_r}{r}\,dr. \quad -(8)$$

The Lagrangian is: $\mathcal{L} = T - U \quad -(9)$

and the Hamiltonian is: $H = T + U \quad -(10)$

where

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad -(11)$$

is the kinetic energy. The Euler Lagrange equation:

2)
$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(12)$$

gives equation (4), and
$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(13)$$

gives the angular momentum:
$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} \quad -(14)$$

with
$$\frac{dL}{dt} = 0. \quad -(15)$$

So L is a constant of motion. The other constant of motion is the hamiltonian (10), so:
$$\frac{dH}{dt} = 0. \quad -(16)$$

In the above, U is the total potential energy:
$$U = -\frac{mMG}{r} + mG\int \frac{\omega_r}{r}\, dr \quad -(17)$$

As in UFT402, eq. (14) gives:
$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(18)$$

so
$$\omega_\phi = 0 \quad -(19)$$

The equation (4) can be transformed into the generalized Binet equation:
$$\boxed{\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}\left(1 + r\omega_r\right)} \quad -(20)$$

which is the vacuum corrected Binet equation.

3) In the limit

$$\omega_r \rightarrow 0 \quad -(21)$$

the solution of eq. (20) is the well known conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(22)$$

Now follow the method of note 403(1) to obtain the precession of the half right latitude. In the limit (21) the half right latitude is:

$$\alpha = r \quad -(23)$$

and is defined by

$$\cos\phi = 0, \quad \phi = \frac{\pi}{2}. \quad -(24)$$

For the conic section (22), the half right latitude is defined by

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) = 0 \quad -(24)$$

so eq. (23) results from the Binet equation:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha} \quad -(25)$$

As in note 403(1), it is assumed that eq. (24) is approximately true for eq. (20), so:

$$\frac{1}{r} \sim \frac{1}{\alpha}\left(1 + r\omega_r\right). \quad -(26)$$

Therefore α / r has increased from 1 at the half right latitude to $1 + r\omega_r$. If the spin connection component ω_r is negative valued, α / r decreases from 1 to $1 - r|\omega_r|$.

The ellipse precesses as shown in Fig (1):

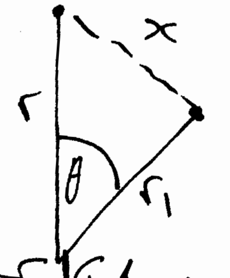

Fig (1)

At the half right latitude:

$$r = \alpha \quad -(27)$$

$$r_1 = \alpha(1 - \alpha|\omega_r|) \quad -(28)$$

By the triangle rule, as in 403(1):

$$x^2 = r^2 + r_1^2 - 2 r r_1 \cos\theta \quad -(29)$$

So

$$r_1 = r\cos\theta \pm \left(r^2\cos^2\theta - (r^2 - x^2)\right)^{1/2}$$

$$\sim r\cos\theta \quad -(30)$$

for small precessions:

$$x \rightarrow 0, \; \cos\theta \rightarrow 1 \quad -(31)$$

so the precession at the half right latitude is:

$$\boxed{\cos\theta \sim \frac{r_1}{r}} \quad -(32)$$

i.e.

$$\boxed{\cos\theta \sim 1 - \alpha|\omega_r|} \quad -(33)$$

Finally use:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \ldots \quad -(34)$$

to find that

$$\boxed{\theta^2 \sim 2\alpha|\omega_r|} \quad -(35)$$

Exact agreement with experimental data is found by adjusting $|\omega_r|$.