Note on Derivative wrt Respect to a Vector

Google "Derivative wrt respect to a vector".

A derivative wrt respect to a vector is the vector gradient, so:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} := \underline{\nabla} \mathcal{L} \quad -(1)$$

so

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} := \frac{\partial \mathcal{L}}{\partial X} \underline{i} + \frac{\partial \mathcal{L}}{\partial Y} \underline{j} + \frac{\partial \mathcal{L}}{\partial Z} \underline{k} \quad -(2)$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} := \frac{\partial \mathcal{L}}{\partial \dot{X}} \underline{i} + \frac{\partial \mathcal{L}}{\partial \dot{Y}} \underline{j} + \frac{\partial \mathcal{L}}{\partial \dot{Z}} \underline{k} \quad -(3)$$

Therefore:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad -(4)$$

means:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{X}} \quad -(5)$$

$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{Y}} \quad -(6)$$

$$\frac{\partial \mathcal{L}}{\partial Z} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{Z}} \quad -(7)$$

So:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \underline{\nabla} \left(\frac{mMG}{r} \right) = mMG \left(\frac{\partial}{\partial X} \left(\frac{1}{(X^2+Y^2)^{1/2}} \right) \underline{i} + \frac{\partial}{\partial Y} \left(\frac{1}{(X^2+Y^2)^{1/2}} \right) \underline{j} \right)$$

$$= -mMG \left(\frac{X}{(X^2+Y^2)^{3/2}} \underline{i} + \frac{Y}{(X^2+Y^2)^{3/2}} \underline{j} \right)$$

$$= -mMG\frac{\underline{r}}{r^3} = \underline{F} = \frac{d\underline{p}}{dt} \quad -(8)$$

Therefore $$\underline{F} = \frac{d\underline{p}}{dt} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\underline{r}}} \quad -(9)$$

which is equation (16) of UFT 377.

Note that Atkins, p. 454 of "Molecular Quantum Mechanics", Appendix 21 of the third edition, uses the equation:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\underline{r}}}\right) = m\ddot{\underline{r}} - e\dot{\underline{A}} - e(\dot{\underline{r}}\cdot\underline{\nabla})\underline{A} \quad -(10)$$

which means:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{X}}\underline{i} + \frac{\partial\mathcal{L}}{\partial\dot{Y}}\underline{j} + \frac{\partial\mathcal{L}}{\partial\dot{Z}}\underline{k}\right)$$

$$= m\ddot{\underline{r}} - e\dot{\underline{A}} - e(\dot{\underline{r}}\cdot\underline{\nabla})\underline{A} \quad -(11)$$