401(5): Final Version of Note 401(4)

Square both sides of Eq. (6) to give:

$$g^2(\text{vac}) = \Omega_0^4 \langle \delta \underline{r} \cdot \delta \underline{r}^* \rangle \quad -(1)$$

From previous work:

$$\underline{F}(\text{vac}) = m \underline{g}(\text{vac}) = \underline{\omega} \phi \quad -(2)$$

and

$$\langle F(\text{vac}) \rangle = \langle F(\text{vac}) \rangle^{(2)} + \langle F(\text{vac}) \rangle^{(4)} + \ldots \quad -(3)$$

To second order:

$$\langle F(\text{vac}) \rangle = \langle F(\text{vac}) \rangle^{(2)} = \frac{1}{6} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \nabla^2 F \quad -(4)$$

It follows that:

$$\begin{aligned} \langle F(\text{vac}) \rangle^2 &= m^2 \Omega_0^4 \langle \delta \underline{r} \cdot \delta \underline{r}^* \rangle \\ &= \omega^2 \phi^2 \\ &= \left(\langle F(\text{vac}) \rangle^{(2)} \right)^2 \\ &= \frac{1}{36} \left(\langle \delta \underline{r} \cdot \delta \underline{r} \rangle \right)^2 \left(\nabla^2 F \right)^2 \end{aligned} \quad -(5)$$

Using

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \langle \delta \underline{r} \cdot \delta \underline{r}^* \rangle \quad -(6)$$

and

$$\phi^2 = \frac{m^2 M^2 G}{r^4} \quad -(7)$$

it follows that

$$\Omega_0^4 = \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle^2 (\nabla^2 F)^2}{36 m^2} \quad -(8)$$

So:

$$\Omega_0^2 = \frac{1}{6m} \langle \underline{\delta r} \cdot \underline{\delta r} \rangle \nabla^2 F \quad - (9)$$

also

$$\nabla^2 F = \frac{4\pi m G}{r^4} \quad - (10)$$

So

$$\Omega_0^2 = \frac{2}{3} m G \frac{\langle \underline{\delta r} \cdot \underline{\delta r} \rangle^{1/2}}{r^4} \quad - (11)$$

<u>This is the same result as Note 401(4).</u>

It also follows that:

$$\omega^2 \phi^2 = \frac{4}{9} \frac{m^2 M^2 G^2}{r^8} \langle \underline{\delta r} \cdot \underline{\delta r} \rangle^2$$

$$= \omega^2 \frac{m^2 M^2 G^2}{r^2} \quad - (12)$$

so

$$\frac{4}{9} \frac{\langle \underline{\delta r} \cdot \underline{\delta r} \rangle^2}{r^6} = \omega^2 \quad - (13)$$

and

$$\frac{2}{3} \frac{\langle \underline{\delta r} \cdot \underline{\delta r} \rangle}{r^3} = \omega \quad - (14)$$

i.e.

$$\boxed{\langle \underline{\delta r} \cdot \underline{\delta r} \rangle = \frac{3}{2} r^3 \omega} \quad - (15)$$