1) 401(1): Conservation of Relativistic Angular Momentum and Forward and Retrograde Precessions

Consider the ECE 2 Lagrangian in plane polar coordinates:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \quad -(1)$$

$$= -mc^2\left(1-\left(\frac{\dot{r}^2+r^2\dot{\phi}^2}{c^2}\right)\right)^{1/2} + \frac{mMG}{r}$$

The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(2)$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(3)$$

Fm eq. (3):

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0. \quad -(4)$$

The relativistic angular momentum is defined as:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(5)$$

and is a constant of motion:

$$\frac{dL}{dt} = 0 \quad -(6)$$

Fm eq. (1):

$$\boxed{L = \gamma m r^2 \dot{\phi}} \quad -(7)$$

Note that:

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2 \quad -(8)$$

so it is clear that relativistic angular momentum is conserved in any coordinate system.

As in UFT377, in the Cartesian system:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \quad -(9)$$

$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}} \quad -(10)$$

From eqs. (1), (9) and (10):

$$\ddot{X} = \frac{mG}{\gamma(X^2+Y^2)^{3/2}}\left(\frac{\dot{X}\dot{Y}Y + X\dot{X}^2}{c^2} - X\right) \quad -(11)$$

$$\ddot{Y} = \frac{mG}{\gamma(X^2+Y^2)^{3/2}}\left(\frac{\dot{Y}\dot{X}X + Y\dot{Y}^2}{c^2} - Y\right) \quad -(12)$$

and these equations produce forward precession.

Note carefully that eqs. (7), (11) and (12) are derived from the same lagrangian (1), so the forward precession conserves relativistic angular momentum, Q.E.D.

In plane polar coordinates eqs. (11) and (12) become eq. (7), and are more equation of motion from eq. (2). This equation of motion is:

3)

$$m \frac{d}{dt}\left(\gamma \dot{r}\right) = -\frac{mMG}{r^2} \quad -(13)$$

$$= m\left(\dot{r}\frac{d\gamma}{dt} + \gamma\ddot{r}\right)$$

where:

$$\gamma = \left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{-1/2} \quad -(14)$$

Therefore the system of equations:

$$\frac{d}{dt}\left(\gamma\dot{r}\right) = -\frac{MG}{r^2}, \quad -(15)$$

and

$$L = \gamma m r^2 \dot{\phi}, \quad -(16)$$

$$\frac{dL}{dt} = 0. \quad -(17)$$

also produces forward precession with conservation of relativistic angular momentum. This has been shown in previous UFT papers. In the classical limit:

$$\gamma \rightarrow 1 \quad -(18)$$

and eq. (13) becomes the Newton equation for orbits, producing a static ellipse or conic section. Retrograde precession is obtained with the same lagrangian (1) written as:

4)
$$\mathcal{L} = -mc^2\left(1 - \frac{\underline{\dot{r}}\cdot\underline{\dot{r}}}{c^2}\right)^{1/2} + \frac{mMG}{|\underline{r}|}$$
$$= -mc^2\left(1 - \left(\frac{\dot{X}^2 + \dot{Y}^2}{c^2}\right)\right)^{1/2} + \frac{mMG}{(X^2+Y^2)^{1/2}}$$
$$= -mc^2\left(1 - \left(\frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)\right)^{1/2} + \frac{mMG}{r} \quad -(19)$$

This is the same lagrangian written in three different ways, so it must conserve relativistic angular momentum.

In order to see how different planetary precessions can emerge from the same lagrangian consider the Hamilton Principle of Least Action:

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \quad -(20)$$

which leads to the Lagrange / Hamilton equations of motion:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \quad -(21)$$

where

$$i = 1, 2, 3 \quad -(22)$$

in three dimensions. These equations are defined for a conservative system with:

$$T = T(\dot{x}_1, \dot{x}_2, \dot{x}_3) \quad -(23)$$
$$U = U(x_1, x_2, x_3) \quad -(24)$$

...) so proper Lagrange variables x_i are defined by eqs (23) and (24). The kinetic energy must be a function of $\dot{x}_i$ and the potential energy must be a function of x_i.

It is well known that there is freedom of choice of the Lagrangian and the proper Lagrange variables. It is this freedom of choice that leads to the fact that the same Lagrangian can produce forward or retrograde precessions.

In quantum field theory for example the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) \quad -(25)$$

leads to the ~~homogeneous~~ field equation of standard electrodynamics. In eq. (25), A_μ is the four potential with:

$$\mu = 0, 1, 2, 3 \quad -(26)$$

The covariant four potential A_μ is a four vector. So the Euler Lagrange equations can be used with vectors as well as scalar components.

For example, if the classical Lagrangian is considered:

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + \frac{mMG}{|\underline{r}|} \quad -(27)$$

the classical momentum is given by:

$$\underline{p} = \frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} = m \underline{\dot{r}} \quad -(28)$$

6) and the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad -(29)$$

give
$$m\ddot{\underline{r}} = -mMG\frac{\underline{r}}{r^3} \quad -(30)$$

which is the Newton equation of orbits, Q.E.D.

In the classic limit the conserved angular momentum is

$$L_0 = mr^2\dot{\phi}, \quad -(31)$$

$$\frac{dL_0}{dt} = 0. \quad -(32)$$

As shown in UFT377, the relativistic lagrangian

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{\underline{r}}\cdot\dot{\underline{r}}}{c^2}\right)^{1/2} + \frac{mMG}{|\underline{r}|} \quad -(33)$$

when used with eq. (29) gives the well known relativistic Newton equation (Marion and Thornton chapter 15):

$$\gamma^3 m\ddot{\underline{r}} = -mMG\frac{\underline{r}}{r^3} \quad -(34)$$

and this gives retrograde precession, a major discovery.

Note carefully that eq. (34) is derived from the same lagrangian (19) as eq. (7), so eq. (34) conserves relativistic angular momentum, Q.E.D, and so does retrograde precession.

) In UFT 377 eq. (34) was worked out in Cartesian
[co]ordinates. It is of interest to demonstrate retrograde
[pr]ecession with plane polar coordinates, in which:

$$\underline{r} = r\,\underline{e}_r \quad -(35)$$

$$\ddot{\underline{r}} = \frac{d^2}{dt^2}(r\,\underline{e}_r) = (\ddot{r} - r\dot{\phi}^2)\,\underline{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\underline{e}_\phi \quad -(36)$$

So eq. (34) becomes:

$$\gamma^3\left((\ddot{r} - r\dot{\phi}^2)\,\underline{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\underline{e}_\phi\right) = -\frac{mG}{r^2}\,\underline{e}_r \quad -(37)$$

So

$$\gamma^3(\ddot{r} - r\dot{\phi}^2) = -\frac{mG}{r^2} \quad -(38)$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(39)$$

$$\gamma = \left(1 - \left(\frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)\right)^{-1/2} \quad -(40)$$

$$L = \gamma m r^2 \dot{\phi} \quad -(41)$$

$$\frac{dL}{dt} = 0 \quad -(42)$$

The system of equations (38) to (42) can be integrated by computer.