398(3): Calculation of $\langle (\delta \underline{r} \cdot \delta \underline{r})^n \rangle$

From previous notes:

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{2}{\pi} \alpha \lambda^2 \log_e \frac{1}{\pi \alpha} = 2.616 \times 10^{-27} m^2 \quad -(1)$$

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle = \frac{4}{3V} (\alpha \lambda^2)^2 \left(\left(\frac{a_0}{\pi} \right)^3 - \left(\frac{\hbar}{mc} \right)^3 \right) \quad -(2)$$

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^3 \rangle = \frac{8\pi \lambda^6 \alpha^3}{6V^3} \left(\left(\frac{a_0}{\pi} \right)^6 - \left(\frac{\hbar}{mc} \right)^6 \right) \quad -(3)$$

Here:

$$\lambda = \frac{\hbar}{mc} = 3.8616 \times 10^{-13} m \quad -(4)$$

$$\alpha = 0.00729735 \quad -(5)$$

$$a_0 = 5.29177 \times 10^{-11} m \quad -(6)$$

$$\frac{a_0}{\pi} = 1.6844 \times 10^{-11} m \quad -(7)$$

So:

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle = \frac{1}{V} \cdot \frac{4}{3} \times 0.0072973^2 \times 3.8616^4 \times 10^{-52} \times 5.29177^3 \times 10^{-33} \quad -(8)$$

to a very good approximation because

$$\frac{a_0}{\pi} \gg \frac{\hbar}{mc} \quad -(9)$$

i.e.
$$\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle = \frac{1.57882 \times 10^{-81}}{V} m^4 \quad -(10)$$

If we assume that the radiation volume is:

$$V = \frac{4}{3}\pi a_0^3 \quad -(11)$$

$$= 6.207 \times 10^{-31}\ m^3$$

where a_0 is the Bohr radius, then

$$\langle (\delta \underline{r} \cdot \delta \underline{r})^2 \rangle = 2.544 \times 10^{-51}\ m^4 \quad -(12)$$

Here a_0 is the Bohr radius.

As in previous UFT papers:

$$\langle r \rangle (1s) = \frac{3}{2} a_0 \quad -(13)$$

$$\langle r \rangle (2s) = 6a_0 \quad -(14)$$

$$\langle r \rangle (3s) = \frac{27}{2} a_0 \quad -(15)$$

and in general the volume V must be worked out from expectation values of the H atom, and the volume element in spherical polar coordinates:

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi \quad -(16)$$

$$= dX\, dY\, dZ$$

The classical volume is

$$V = \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \quad -(17)$$

$$= \frac{4}{3}\pi R^3$$

In the H atom:

$$\langle r \rangle = \int \psi^* r \psi \, d\tau \quad -(18)$$

So the expectation value of V is:

$$\langle V \rangle = \frac{4}{3}\pi \int \phi^* r^3 \phi \, d\tau \quad -(19)$$

for each wavefunction. The normalization condition is:

$$\int \phi^* \phi \, d\tau = 1 \quad -(20)$$

The basic assumption of this theory is that

$$m \frac{d^2 \delta \underline{r}}{dt^2} = -e\underline{E} \quad -(21)$$

as in Note 398(2). It leads to Eqs. (1) to (3), Eqs. (2) and (3) being classical. The smaller the volume V, the larger the corrections (2) and (3), so in a nucleus for example, these corrections can be important.