Squeezing More from the Tetrad Postulate

for an Antisymmetric Torsion

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Torsion-Metric Equation

For the totally antisymmetric torsion, it was found in UFT354 and PECE2 that from metric compatibility,

$$\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2}\, g^{\rho\sigma}\left(\frac{\partial(g_{\nu\sigma}+g_{\sigma\nu})}{\partial x^{\mu}} + \frac{\partial(g_{\sigma\mu}+g_{\mu\sigma})}{\partial x^{\nu}} - \frac{\partial(g_{\mu\nu}+g_{\nu\mu})}{\partial x^{\sigma}} - 2\,T_{\nu\mu\sigma} - 2\,T_{\mu\nu\sigma} + T_{\sigma\mu\nu} + T_{\sigma\nu\mu}\right) \tag{1}$$

which reduces to the following upon the assumption of total antisymmetry of torsion, and symmetry of the metric,

$$\Gamma^{\rho}{}_{\nu\mu} = g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) - \Gamma^{\rho}{}_{\mu\nu}. \tag{2}$$

The definition of torsion is

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu}. \tag{3}$$

Combining (2) and (3) gives

$$T^{\rho}{}_{\mu\nu} = 2\,\Gamma^{\rho}{}_{\mu\nu} - g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right). \tag{4}$$

Antisymmetry of the torsion tensor requires that when $\mu=\nu$ torsion vanishes, so that

$$\Gamma^{\rho}{}_{\mu\mu} = \frac{g^{\rho\sigma}}{2}\left(2\,\frac{\partial g_{\mu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\mu}}{\partial x^{\sigma}}\right)\ . \tag{5}$$

Note that equation (4) can be cast in form notation as (torsion-metic equation)

$$T^{a}{}_{\mu\nu} = 2\,\Gamma^{\rho}{}_{\mu\nu}\, q^{a}_{\rho} - q^{a}_{\rho}\, g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right). \tag{6}$$

Tetrad Postulate

From the tetrad postulate,

$$\partial_{\mu}\, q^{a}_{\nu} + \omega^{a}{}_{\mu b}\, q^{b}_{\nu} - \Gamma^{\rho}{}_{\mu\nu}\, q^{a}_{\rho} = 0 \tag{7}$$

therefore (6) to be rewritten

$$T^{a}{}_{\mu\nu} = 2\ \partial_{\mu}\, q^{a}_{\nu} + 2\,\omega^{a}{}_{\mu b}\, q^{b}_{\nu} - q^{a}_{\rho}\, g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right). \tag{8}$$

Antisymmetry of the torsion tensor then gives

$$2\ \partial_{\mu}\, q^{a}_{\nu} + 2\,\omega^{a}{}_{\mu b}\, q^{b}_{\nu} - q^{a}_{\rho}\, g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) = -\left(2\ \partial_{\nu}\, q^{a}_{\mu} + 2\,\omega^{a}{}_{\nu b}\, q^{b}_{\mu} - q^{a}_{\rho}\, g^{\rho\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}}\right)\right)$$

which simplifies to

$$\partial_{\mu}\, q^{a}_{\nu} + \omega^{a}{}_{\mu b}\, q^{b}_{\nu} + \partial_{\nu}\, q^{a}_{\mu} + \omega^{a}{}_{\nu b}\, q^{b}_{\mu} = q^{a}_{\rho}\, g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right). \tag{9}$$

Because of the antisymmetry conditions we have

$$q^{a}_{\rho}\, g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) = 0\ . \tag{10}$$

Since this is true for any q^{a}_{ρ}, we must have

$$g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) = 0\ . \tag{11}$$

It is interesting to note the case of zero torsion, ie. standard Einstein gravitation, which follows from equation (4)

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\, g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) \tag{12}$$

Equation (12) can only be true if, given equation (11),

$$\Gamma^{\rho}{}_{\mu\nu} = 0 \tag{13}$$

that is, a flat spacetime, or

$$\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} = 0 \tag{14}$$

a torsion free spacetime.

We can say that metric compatibility therefore, valid for spacetimes that are flat, curved only, or curved with torsion. This was deduced in PECE2 Chapter 2, without the use of the antisymmetry condition.

Trace of the Gamma Connection

The trace(s) of the gamma connection in tensorial notation can be written (summed over repeated indices)

$$\Gamma^{\rho}_{\mu\mu} = \Gamma^{\rho}_{\mu\nu}\ \delta^{\nu}_{\mu} = 0 \tag{15}$$

$$\Gamma^{\mu}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}\ \delta^{\mu}_{\rho} = 0 \tag{16}$$

$$\Gamma^{\nu}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}\ \delta^{\nu}_{\rho} = 0\ . \tag{17}$$

The sums are all assumed to be zero following the lead of paper 298., but this needs to be proven. This represents twelve equations, four each for of equations (14) through (16). They are not all independent, for example. if one applies the antisymmetry of $\Gamma^{\rho}_{\mu\nu}$ to equation (16), we have

$$\Gamma^{\nu}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}\ \delta^{\nu}_{\rho} = -\,\Gamma^{\rho}_{\nu\mu}\,\delta^{\nu}_{\rho} = -\Gamma^{\nu}{}_{\nu\mu}\ . \tag{18}$$

Equations (15) and (16) are the same (since it doesn't matter what the summation index is called). It is interesting to note the antisymmetry of the two traces $\Gamma^{\nu}_{\mu\nu}$ and $\Gamma^{\nu}_{\nu\mu}$ in equation (17). This holds even if the torsion is not totally antisymmetric.

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