395(2): Effect of the Vacuum on Electron / Electron Spin Spin Interaction

On the classical level, the interaction of two magnetic dipole moments $\underline{m}_1$ and $\underline{m}_2$ is given by:

$$E_n = \frac{\mu_0}{4\pi r^3}\left(\underline{m}_1 \cdot \underline{m}_2 - 3\underline{m}_1 \cdot \hat{\underline{r}}\,\hat{\underline{r}} \cdot \underline{m}_2\right) \quad -(1)$$

where

$$\hat{\underline{r}} = \underline{r} \,/\, |\underline{r}| \quad -(2)$$

Consider the interaction of two electrons of charge e and mass m, then:

$$\underline{m}_1 = \frac{e}{m}\underline{S}_1 \; ; \; \underline{m}_2 = \frac{e}{m}\underline{S}_2 \quad -(3)$$

where $\underline{S}_1$ and $\underline{S}_2$ are the spin angular momenta of the electrons. The interaction hamiltonian can be expressed as:

$$H^{(1)} = -\underline{m}_1 \cdot \underline{B}(\underline{m}_2) \quad -(4)$$

where $\underline{B}$ is the magnetic flux density due to the magnetic dipole potential of the second electron:

$$\underline{A} = \frac{\mu_0}{4\pi r^3}\,\underline{m}_2 \times \underline{r} \quad -(5)$$

In the absence of the vacuum the magnetic field $\underline{B}(\underline{m}_2)$ consists of the dipole term:

$$\underline{B}_D(\underline{m}_2) = \frac{\mu_0}{4\pi r^3}\left(3\frac{\underline{r}\,\underline{r}}{r^2} \cdot \underline{m}_2 - \underline{m}_2\right) - (6)$$

and the contact term:

$$\underline{B}_c(\underline{m}_2) = -\frac{\mu_0}{4\pi}\,\underline{m}_2 \nabla^2\left(\frac{1}{r}\right) - (7)$$

The dipole term leads to a hamiltonian:

2)

$$H_D^{(1)}\psi = En_D\psi \quad -(8)$$

and the contact term to:

$$H_c^{(1)}\psi = En_c\psi \quad -(9)$$

Here

$$H_D^{(1)}\psi = \frac{\mu_0}{4\pi r^3}\frac{e^2}{m^2}\left(\hat{\underline{S}}_1\cdot\hat{\underline{S}}_2 - 3\hat{\underline{S}}_1\cdot\hat{\underline{r}}\,\hat{\underline{r}}\cdot\hat{\underline{S}}_2\right)\psi \quad -(10)$$

The usual procedure is to consider $\hat{\underline{S}}_1$ and $\hat{\underline{S}}_2$ aligned in the Z axis, so:

$$H_D^{(1)}\psi = \frac{\mu_0 e^2}{4\pi m^2 r^3}\left(1-3\cos^2\theta\right)S_{1Z}S_{2Z}\psi \quad -(11)$$

which follows from:

$$\cos\theta = \frac{Z}{r} \quad -(12)$$

in spherical polar coordinates.

In quantum mechanics:

$$\hat{S}_Z\psi = \hbar m_s\psi \quad -(13)$$

where

$$m_s = \pm\frac{1}{2}. \quad -(14)$$

So

$$H_D^{(1)}\psi = \frac{\mu_0\hbar^2 e^2}{4\pi m^2 r^3}\left(1-3\cos^2\theta\right)m_{1s}m_{2s}\psi, \quad -(15)$$

$$m_{1s} = \pm\frac{1}{2}, \quad m_{2s} = \pm\frac{1}{2} \quad -(16)$$

The units of eq. (15) are correct because:

$\mu_0 = J s^2 C^{-2} m^{-1}$, so $H_D^{(1)} = J s^2 C^{-2} m^{-1} J^2 s^2 C^{-2} / kg^2 m^3 = J^3 / (kg^2 m^4 s^{-4}) = J$ ✓

) The Hamiltonian of the contact term gives the energy:

$$H_c^{(1)} \psi = E_c \psi \quad -(17)$$

$$= \frac{\mu_0 \hbar^2 e^2}{4\pi m^2} \nabla^2 \left(\frac{1}{r}\right) m_{1S} m_{2S} \psi$$

Using the result:

$$\nabla^2 \left(\frac{1}{r}\right) = 0 \quad -(18)$$

the contact term gives no contribution. Eq. (18) is the result of ordinary differentiation.

However in the standard model, the Dirac delta function is introduced:

$$\underline{B}_c = -\frac{\mu_0}{4\pi} \underline{m} \nabla^2 \left(\frac{1}{r}\right) = \mu_0 \underline{m} \, \delta_D(\underline{r}) \quad -(19)$$

so $$H_c^{(1)} = -\mu_0 \underline{m}_1 \cdot \underline{m}_2 \, \delta_D(\underline{r}) \quad -(20)$$

and $$H_c^{(1)} \psi = E_c^{(1)} \psi \quad -(21)$$

The contact interaction energy is: $\quad -(22)$

$$E_c^{(1)} = \mu_0 \frac{e^2 \hbar^2}{m^2} m_1 m_2 \int \psi^* \delta_D(\underline{r}) \psi.$$

The standard model proceeds by using the fact that the integral in equation (22) is not zero for certain types of wave function. This is the basis for the usual Land? calculation. The next note will modify these calculations for the effect of the vacuum.