394(2): Application of Antisymmetry to the Travelling Electric and Magnetic Dipole Fields.

The electric dipole field in the presence of the vacuum is:

$$\underline{E}(\underline{r}+\delta\underline{r}) = \frac{1}{4\pi\epsilon_0|\underline{r}+\delta\underline{r}|^3}\left(\frac{(\underline{r}+\delta\underline{r})(\underline{p}\cdot(\underline{r}+\delta\underline{r}))}{|\underline{r}+\delta\underline{r}|^2} - \underline{p}\right)$$

$$= \underline{E}_0(\underline{r}) + \underline{\omega}(\underline{r}+\delta\underline{r})\,\phi_0(\underline{r}) \quad -(1)$$

where

$$E_0 = \frac{1}{4\pi\epsilon_0 r^3}\left(\frac{\underline{r}\,\underline{p}\cdot\underline{r}}{r^2} - \underline{p}\right) \quad -(2)$$

and

$$\phi_0 = \frac{1}{4\pi\epsilon_0 r^3}\,\underline{r}\cdot\underline{p} \quad -(3)$$

The experimentally observed field is:

$$\underline{E}(\mathrm{obs}) = \langle \underline{E}(\underline{r}+\delta\underline{r})\rangle \quad -(4)$$

and the ensemble averaged spin connection is $\langle \underline{\omega}(\underline{r}+\delta\underline{r})\rangle$

The scalar antisymmetry law is:

$$\underline{E} = \underline{E}_0 + \underline{\omega}\,\phi_0 = -\frac{\partial \underline{A}_E}{\partial t} - \omega_0 \underline{A}_E \quad -(5)$$

where ω_0 is the scalar spin connection and where $\underline{A}_E$ is the electric vector potential.

The dipole moment is:

$$\underline{p} = \int \underline{x}'\rho(\underline{x}')\,d^3x' \quad -(6)$$

) and integrate out the dependence on x'. The dipole moment is an intrinsic property of a molecule. In general, the scalar potential is:

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} d^3x' \quad -(7)$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

in a well known spherical harmonic expansion.

For an electric dipole along the Z axis:

$$\underline{E}_0 = \frac{p}{4\pi\epsilon_0 r^3} \left(2\cos\theta \, \underline{e}_r + \sin\theta \, \underline{e}_\theta \right) \quad -(8)$$

and

$$\underline{E} = \frac{p}{4\pi\epsilon_0 |\underline{r}+\delta\underline{r}|^3} \left(2\cos\theta \, \underline{e}_r + \sin\theta \, \underline{e}_\theta \right). \quad -(9)$$

It follows that:

$$\langle \underline{E} \rangle = \underline{E}_0 + \langle \underline{\omega} \rangle \phi_0 \quad -(10)$$

where

$$\langle \underline{E} \rangle = \frac{p}{4\pi\epsilon_0 r^3} \left(1 - \frac{3}{2} \frac{\langle \delta\underline{r} \cdot \delta\underline{r} \rangle}{r^2} \left(2\cos\theta \, \underline{e}_r + \sin\theta \, \underline{e}_\theta \right) \right) \quad -(11)$$

For a dipole in the Z axis:

$$\phi_0 = \frac{1}{4\pi\epsilon_0 r^3} \underline{r} \cdot \underline{p} = \frac{Z p}{4\pi\epsilon_0 r^3} \quad -(12)$$

where

$$Z = r\cos\theta \quad -(13)$$

so

$$\phi_0 = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \quad -(14)$$

It follows that:

$$\frac{p}{4\pi\epsilon_0 r^3}\left(1-\frac{3}{2}\frac{\langle\delta\underline{r}\cdot\delta\underline{r}\rangle}{r^2}\right)\left(2\cos\theta\,\underline{e}_r+\sin\theta\,\underline{e}_\theta\right) \quad -(15)$$

$$= \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\underline{e}_r+\sin\theta\,\underline{e}_\theta\right)+\frac{\langle\underline{\omega}\rangle p\cos\theta}{4\pi\epsilon_0 r^2}$$

So:

$$\langle\omega_r\rangle = -3\frac{\langle\delta\underline{r}\cdot\delta\underline{r}\rangle}{r^3} \quad -(16)$$

$$\langle\omega_\theta\rangle = -\frac{3}{2}\frac{\langle\delta\underline{r}\cdot\delta\underline{r}\rangle}{r^3}\tan\theta \quad -(17)$$

Eqs. (16) and (17) are ensemble averaged vacuum maps, i.e. averaged spin connections.

The magnetic dipole field in general is:

$$\underline{B}_0(\underline{x}) = \underline{\nabla}\times\underline{A}_0(\underline{x}), \quad -(18)$$

so the vacuum corrected magnetic flux density is:

$$\underline{B} = \underline{B}_0 - \underline{\omega}\times\underline{A}_0 \quad -(19)$$

and

$$\langle\underline{B}(\underline{x}+\delta\underline{x})\rangle = B_0(\underline{x}) - \langle\underline{\omega}(x+\delta\underline{x})\rangle\times\underline{A}_0(\underline{x}) \quad -(20)$$

4) In general:

$$\underline{A}_0(\underline{x}) = \frac{\mu_0}{4\pi}\int \frac{\underline{J}_0(\underline{x}')}{|\underline{x}-\underline{x}'|} d^3x' \quad -(21)$$

where $\underline{J}_0$ is the matter current density in the hypothetical absence of the vacuum. In the dipole approximation:

$$\underline{A}_0(\underline{x}) = \frac{\mu_0}{4\pi}\frac{\underline{m}\times\underline{x}}{|\underline{x}|^3} \quad -(22)$$

where the magnetic dipole moment is:

$$\underline{m} = \frac{1}{2}\int \underline{x}'\times\underline{J}(\underline{x}')\, d^3x' \quad -(23)$$

So

$$\underline{B}_0(\underline{x}) = \frac{\mu_0}{4\pi}\left(\frac{3\underline{x}(\underline{x}\cdot\underline{m})}{|\underline{x}|^5} - \frac{\underline{m}}{|\underline{x}|^3}\right) \quad -(24)$$

and:

$$\underline{B}(\underline{x}+\delta\underline{x}) = \frac{\mu_0}{4\pi}\left(\frac{3(\underline{x}+\delta\underline{x})(\underline{x}+\delta\underline{x})\cdot\underline{m}}{|\underline{x}+\delta\underline{x}|^5} - \frac{\underline{m}}{|\underline{x}+\delta\underline{x}|^3}\right) \quad -(25)$$

The vector antisymmetry law applies to:

$$\langle\underline{B}\rangle = \underline{\nabla}\times\underline{A}_0 - \langle\underline{\omega}\rangle\times\underline{A}_0 \quad -(26)$$

so

$$\frac{\partial A_{Z0}}{\partial Y} + \frac{\partial A_{Y0}}{\partial Z} = \langle\omega_Y\rangle A_{Z0} + \langle\omega_Z\rangle A_{Y0}$$

$$\frac{\partial A_{X0}}{\partial Z} + \frac{\partial A_{Z0}}{\partial X} = \langle\omega_Z\rangle A_{X0} + \langle\omega_X\rangle A_{Z0}$$

$$\frac{\partial A_{Y0}}{\partial X} + \frac{\partial A_{X0}}{\partial Y} = \langle\omega_X\rangle A_{Y0} + \langle\omega_Y\rangle A_{X0}$$

$$-(27)$$

5) Therefore the averaged spin connections can be worked out from eqs. (27).

Using the magnetic dipole fields far from a current loop, as in the previous note:

$$\langle \underline{B} \rangle = \frac{\mu_0 I \pi a^2}{4\pi r^3}\left(1 - \frac{3}{2r^2}\langle \delta\underline{r} \cdot \delta\underline{r} \rangle\right)\left(\cos\theta \, \underline{e}_r + \sin\theta \, \underline{e}_\theta\right)$$

$$= \underline{B}_0 - \langle \underline{\omega} \rangle \times \underline{A}_0 \quad - (27)$$

where:

$$\underline{B}_0 = \frac{\mu_0 I \pi a^2}{4\pi r^3}\left(\cos\theta \, \underline{e}_r + \sin\theta \, \underline{e}_\theta\right) \quad - (28)$$

and

$$\underline{A}_0 = \frac{\mu_0 I a^2}{4\pi r^2} \sin\theta \, \underline{e}_\phi \quad - (29)$$

so

$$\frac{\langle \delta\underline{r} \cdot \delta\underline{r} \rangle}{r^5}\left(\cos\theta \, \underline{e}_r + \sin\theta \, \underline{e}_\theta\right) = \frac{\langle \underline{\omega} \rangle \times \underline{e}_\phi}{r^2} \quad - (30)$$

Here:

$$\underline{e}_r = \sin\theta\cos\phi \, \underline{i} + \sin\theta\sin\phi \, \underline{j} + \cos\theta \, \underline{k}$$

$$\underline{e}_\theta = \cos\theta\cos\phi \, \underline{i} + \cos\theta\sin\phi \, \underline{j} - \sin\theta \, \underline{k}$$

$$\underline{e}_\phi = -\sin\phi \, \underline{i} + \cos\phi \, \underline{j} \quad - (31)$$

so:

$$\underline{e}_r \times \underline{e}_\theta = \underline{e}_\phi$$

$$\underline{e}_\phi \times \underline{e}_r = \underline{e}_\theta \quad - (32)$$

$$\underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r$$

and:

$$\frac{\langle \underline{\delta r} \cdot \underline{\delta r} \rangle}{r^3}\left(\cos\theta\, \underline{e}_r + \sin\theta\, \underline{e}_\theta\right) = \left(\langle \omega_r \rangle \underline{e}_r + \langle \omega_\theta \rangle \underline{e}_\theta\right) \times \underline{e}_\phi \quad -(33)$$

So

$$\langle \omega_r \rangle = -\frac{\langle \underline{\delta r} \cdot \underline{\delta r} \rangle}{r^3} \cos\theta \quad -(34)$$

$$\langle \omega_\theta \rangle = -\frac{\langle \underline{\delta r} \cdot \underline{\delta r} \rangle}{r^3} \sin\theta \quad -(35)$$

so

$$\langle \underline{\omega} \rangle = -\frac{\langle \underline{\delta r} \cdot \underline{\delta r} \rangle}{r^3}\left(\cos\theta\, \underline{e}_r + \sin\theta\, \underline{e}_\phi\right) \quad -(36)$$

and

$$\underline{A}_0 = \frac{\mu_0 I a^2}{4\pi r^2} \sin\theta\, \underline{e}_\phi \quad -(37)$$

The vector antisymmetry law in this case must be constructed from:

$$\langle \underline{B} \rangle = \underline{B}_0 - \langle \underline{\omega} \rangle \times \underline{A}_0 \quad -(38)$$

$$= \underline{\nabla} \times \underline{A}_0 - \langle \underline{\omega} \rangle \times \underline{A}_0$$

Antisymmetry means:

$$\langle B \rangle_{ij} = -\langle B \rangle_{ji} \quad -(39)$$

where

$$\langle B \rangle_k = \epsilon_{ijk} \langle B \rangle_{ij} \quad -(40)$$

So eq. (38) is automatically antisymmetric:

$$\langle \underline{B} \rangle = \underline{B}_0 + \underline{B}_{vac} \quad -(41) \quad -(42)$$

and

$$(\underline{B}_0)_{ij} = -(\underline{B}_0)_{ji} \; ; \; (\underline{B}_{vac})_{ij} = -(\underline{B}_{vac})_{ji}$$