393(8): Final Expression for the Electric Field Strength $\underline{E}$.

The dipole electric field strength in the presence of the vacuum is:

$$\underline{E} = \frac{1}{4\pi\epsilon_0 r^5}(\underline{r}+\delta\underline{r})(\underline{p}\cdot(\underline{r}+\delta\underline{r}))\left(1-\frac{5x}{2}\right) - \frac{\underline{p}}{4\pi\epsilon_0 r^3}\left(1-\frac{3x}{2}\right) \quad -(1)$$

where

$$x = \frac{1}{r^2}(2\underline{r}\cdot\delta\underline{r} + \delta\underline{r}\cdot\delta\underline{r}) \quad -(2)$$

So:

$$\underline{E} = \frac{1}{4\pi\epsilon_0 r^5}\left((\underline{r}+\delta\underline{r})\underline{p}\cdot(\underline{r}+\delta\underline{r}) - \frac{5x}{2}(\underline{r}+\delta\underline{r})(\underline{p}\cdot(\underline{r}+\delta\underline{r}))\right)$$

$$- \frac{\underline{p}}{4\pi\epsilon_0 r^3} + \frac{3x\underline{p}}{8\pi\epsilon_0 r^3}$$

$$= \frac{1}{4\pi\epsilon_0 r^5}(\underline{r}\,\underline{p}\cdot\underline{r} + \delta\underline{r}\,\underline{p}\cdot\underline{r} + \underline{r}\,\underline{p}\cdot\delta\underline{r} + \delta\underline{r}\,\underline{p}\cdot\delta\underline{r})\left(1-\frac{5x}{2}\right)$$

$$- \frac{\underline{p}}{4\pi\epsilon_0 r^3}\left(1-\frac{3x}{2}\right) \quad -(2)$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left(\frac{\underline{r}\,\underline{p}\cdot\underline{r}}{r^2} - \underline{p}\right) - \frac{x}{8\pi\epsilon_0 r^3}\left(\frac{5\underline{r}\,\underline{p}\cdot\underline{r}}{r^2} - 3\underline{p}\right)$$

$$+ \frac{1}{4\pi\epsilon_0 r^5}(\delta\underline{r}\,\underline{p}\cdot\underline{r} + \underline{r}\,\underline{p}\cdot\delta\underline{r} + \delta\underline{r}\,\underline{p}\cdot\delta\underline{r})\left(1-\frac{5x}{2}\right)$$

As shown in Note 393(7):

$$\langle x \rangle = \quad -(3)$$

$$= \frac{1}{r^2}\langle(2\underline{r}\cdot\delta\underline{r} + \delta\underline{r}\cdot\delta\underline{r})\rangle$$

$$= \frac{1}{r^2}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle$$

) in an isotropic sample, where:

$$\langle \delta\underline{r} \rangle = \underline{0} \quad -(4)$$

Therefore:

$$\langle \underline{E} \rangle = \underline{E}_0 - \frac{\langle \delta\underline{r} \cdot \delta\underline{r} \rangle}{8\pi\epsilon_0 r^5}\left(\frac{5\underline{r}\,\underline{p}\cdot\underline{r}}{r^2} - 3\underline{p} \right)$$

$$+ \frac{1}{4\pi\epsilon_0 r^5}\left\langle \delta\underline{r}\,\underline{p}\cdot\underline{r} + \underline{r}\,\underline{p}\cdot\delta\underline{r} + \delta\underline{r}\,\underline{p}\cdot\delta\underline{r}\left(1 - \frac{5x}{2}\right)\right\rangle \quad -(5)$$

where

$$\underline{E}_0 = \frac{1}{4\pi\epsilon_0 r^3}\left(\frac{\underline{r}\,\underline{p}\cdot\underline{r}}{r^2} - \underline{p} \right) \quad -(6)$$

is the standard model dipole field, the field in the hypothetical absence of the vacuum. From observations of the Lamb shift it is known that:

$$\frac{5x}{2} \ll 1 \quad -(7)$$

so:

$$\langle \underline{E}_1 \rangle = \frac{1}{4\pi\epsilon_0 r^5}\left(\langle \delta\underline{r}\,\underline{p}\cdot\underline{r} \rangle + \langle \underline{r}\,\underline{p}\cdot\delta\underline{r} \rangle + \langle \delta\underline{r}\,\underline{p}\cdot\delta\underline{r} \rangle \right) \quad -(8)$$

to an excellent approximation

By isotropy:

$$\langle \delta\underline{r}\,\underline{p}\cdot\underline{r} \rangle = \langle \underline{r}\,\underline{p}\cdot\delta\underline{r} \rangle = \underline{0} \quad -(9)$$

3) So:

$$\langle \underline{E_1} \rangle = \frac{1}{4\pi\epsilon_0 r^5} \langle \delta\underline{r}\, \underline{p} \cdot \delta\underline{r} \rangle \quad -(10)$$

$$= \frac{1}{4\pi\epsilon_0 r^5} \left\langle (p_x \delta x + p_y \delta y + p_z \delta z)(\delta x \underline{i} + \delta y \underline{j} + \delta z \underline{k}) \right\rangle$$

By isotropy:

$$\langle \delta x \delta y \rangle = \langle \delta x \delta z \rangle = \langle \delta y \delta z \rangle = 0 \quad -(11)$$

So

$$\langle \underline{E}_1 \rangle = \frac{1}{4\pi\epsilon_0 r^5} \langle p_x \delta x^2 \underline{i} + p_y \delta y^2 \underline{j} + p_z \delta z^2 \underline{k} \rangle$$

$$= \frac{1}{4\pi\epsilon_0 r^5} \left(p_x \underline{i} \langle \delta x^2 \rangle + p_y \underline{j} \langle \delta y^2 \rangle + p_z \underline{k} \langle \delta z^2 \rangle \right) \quad -(12)$$

By isotropy:

$$\langle \delta x^2 \rangle = \langle \delta y^2 \rangle = \langle \delta z^2 \rangle \quad -(13)$$

and

$$\langle \delta r^2 \rangle = \langle \delta x^2 \rangle + \langle \delta y^2 \rangle + \langle \delta z^2 \rangle \quad -(14)$$

So

$$\langle \delta r^2 \rangle = 3 \langle \delta x^2 \rangle = 3 \langle \delta y^2 \rangle = 3 \langle \delta z^2 \rangle \quad -(15)$$

and

$$\langle \delta x^2 \rangle = \langle \delta y^2 \rangle = \langle \delta z^2 \rangle = \frac{1}{3} \langle \delta r^2 \rangle$$

$$= \frac{1}{3} \langle \delta\underline{r} \cdot \delta\underline{r} \rangle \quad -(16)$$

It follows that:

$$\langle \underline{E}_1 \rangle = \frac{\langle \delta\underline{r} \cdot \delta\underline{r} \rangle}{4\pi\epsilon_0 r^5} \underline{p} \quad -(17)$$

4) Finally:

$$\langle \underline{E} \rangle = \underline{E}_0 - \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{8\pi\epsilon_0 r^5} \left(\frac{5 \underline{r}\, \underline{p} \cdot \underline{r}}{r^2} - 3\underline{p} \right) + \frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{4\pi\epsilon_0 r^5} \underline{p} \quad -(18)$$

$$\boxed{\langle \underline{E} \rangle = \underline{E}_0 - \frac{5 \langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{8\pi\epsilon_0 r^5} \left(\frac{\underline{r}\, \underline{p} \cdot \underline{r}}{r^2} - \underline{p} \right)} \quad -(19)$$

It is seen that the correction due to the vacuum is proportional to $\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$.

If the approximation (7) is not used, other terms appear in eq. (19), and these are worked out in Section 3 of UFT393.