393(6): The Electric Dipole Field in the Presence of the Vacuum

From previous notes the electric dipole field is:

$$\underline{E} = \frac{1}{4\pi\epsilon_0 r^5}(\underline{r}+\delta\underline{r})(\underline{p}\cdot(\underline{r}+\delta\underline{r}))\left(1-\frac{5x}{2}\right)$$

$$-\frac{\underline{p}}{4\pi\epsilon_0 r^3}\left(1-\frac{3x}{2}\right) \quad -(1)$$

where:

$$x = \frac{1}{r^2}(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r}) \quad -(2)$$

The ensemble averaged $\underline{E}$ is:

$$\langle\underline{E}\rangle = \frac{1}{4\pi\epsilon_0 r^5}\langle(\underline{r}+\delta\underline{r})(\underline{p}\cdot(\underline{r}+\delta\underline{r}))\rangle$$

$$-\frac{5}{8\pi\epsilon_0 r^5}\langle(\underline{r}+\delta\underline{r})(\underline{p}\cdot(\underline{r}+\delta\underline{r}))(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r})\rangle$$

$$-\frac{\underline{p}}{4\pi\epsilon_0 r^3}+\frac{3\underline{p}}{8\pi\epsilon_0 r^3}\langle x\rangle \quad -(3)$$

In eq. (3):

$$\langle x\rangle = \frac{1}{r^2}(\langle 2\underline{r}\cdot\delta\underline{r}\rangle+\langle\delta\underline{r}\cdot\delta\underline{r}\rangle) \quad -(4)$$

$$= \frac{1}{r^2}(2\underline{r}\cdot\langle\delta\underline{r}\rangle+\langle\delta\underline{r}\cdot\delta\underline{r}\rangle)$$

$$= \frac{1}{r^2}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle$$

because by isotropy:

$$\langle\delta\underline{r}\rangle = \underline{0} \quad -(5)$$

a) Consider:

$$\left\langle (\underline{r}\cdot\underline{p})(\underline{r}+\delta\underline{r}) + (\underline{p}\cdot\delta\underline{r})(\underline{r}+\delta\underline{r}) \right\rangle$$

$$= (\underline{r}\cdot\underline{p})\underline{r} + \underline{r}\cdot\underline{p}\,\langle\delta\underline{r}\rangle + \langle(\underline{p}\cdot\delta\underline{r})\underline{r}\rangle + \langle(\underline{p}\cdot\delta\underline{r})\delta\underline{r}\rangle \quad -(6)$$

$$= (\underline{r}\cdot\underline{p})\underline{r} + \langle(\underline{p}\cdot\delta\underline{r})\underline{r}\rangle + \langle(\underline{p}\cdot\delta\underline{r})\delta\underline{r}\rangle$$

Expanding in Cartesian coordinates:

$$\langle(\underline{p}\cdot\delta\underline{r})\underline{r}\rangle = \langle(p_x\delta x + p_y\delta y + p_z\delta z)(x\underline{i} + y\underline{j} + z\underline{k})\rangle$$

$$= p_x x\underline{i}\,\langle\delta x\rangle + \cdots$$

$$= \underline{0} \quad -(7).$$

$$\langle(\underline{p}\cdot\delta\underline{r})\delta\underline{r}\rangle = \langle(p_x\delta x + p_y\delta y + p_z\delta z)\cdot(\delta x\,\underline{i} + \delta y\,\underline{j} + \delta z\,\underline{k})\rangle \quad -(8)$$

$$= \underline{i}\,p_x\langle\delta x^2\rangle + \cdots$$

By isotropy:

$$\langle\delta x\,\delta y\rangle = \langle\delta x\,\delta z\rangle = \langle\delta y\,\delta z\rangle = 0 \quad -(9)$$

and

$$\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle = \frac{1}{3}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle \quad -(10)$$

so

$$\langle(\underline{p}\cdot\delta\underline{r})\delta\underline{r}\rangle = \frac{1}{3}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\,\underline{p} \quad -(11)$$

Therefore in eq. (3):

$$\langle (\underline{r}+\delta\underline{r})(\underline{p}\cdot(\underline{r}+\delta\underline{r}))\rangle$$

$$= (\underline{r}\cdot\underline{p})\underline{r} + \frac{1}{3}\langle \delta\underline{r}\cdot\delta\underline{r}\rangle \underline{p} \quad -(12)$$

Therefore:

$$\underline{E} = \frac{1}{4\pi\epsilon_0 r^5}\left((\underline{r}\cdot\underline{p})\underline{r} + \frac{1}{3}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\underline{p}\right) \quad -(13)$$

$$- \frac{\underline{p}}{4\pi\epsilon_0 r^3} + \frac{3\underline{p}}{8\pi\epsilon_0 r^5}\langle\delta\underline{r}\cdot\delta\underline{r}\rangle - \underline{E}_1$$

where:

$$\underline{E}_1 = \frac{5}{8\pi\epsilon_0 r^5}\left\langle(\underline{r}+\delta\underline{r})(\underline{p}\cdot(\underline{r}+\delta\underline{r}))(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r})\right\rangle \quad -(14)$$

So:

$$\underline{E} = \frac{1}{4\pi\epsilon_0 r^3}\left(\frac{(\underline{r}\cdot\underline{p})\underline{r}}{r^2} - \underline{p}\right) + \frac{\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\underline{p}}{4\pi\epsilon_0}\left(\frac{1}{3r^5} + \frac{3}{2}\frac{\langle\delta\underline{r}\cdot\delta\underline{r}\rangle}{r^5}\right) - \underline{E}_1$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left(\frac{(\underline{r}\cdot\underline{p})\underline{r}}{r^2} - \underline{p}\right) + \frac{\langle\delta\underline{r}\cdot\delta\underline{r}\rangle}{4\pi\epsilon_0 r^5}\left(\frac{1}{3}+\frac{3}{2}\right)\underline{p} - \underline{E}_1$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left(\frac{(\underline{r}\cdot\underline{p})\underline{r}}{r^2} - \underline{p}\right) + \frac{11}{24}\frac{\langle\delta\underline{r}\cdot\delta\underline{r}\rangle\underline{p}}{4\pi\epsilon_0 r^5} - \underline{E}_1 \quad (15)$$

The first term on the right hand side is the electric dipole field in the hypothetical absence of the vacuum.

4) In eq. (15):

$$\underline{E}_1 = \frac{1}{r^2}\left\langle (\underline{r}+\delta\underline{r})(\underline{p}\cdot(\underline{r}+\delta\underline{r}))(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r})\right\rangle$$

$$= \frac{1}{r^2}\Big\langle \underline{r}(\underline{p}\cdot(\underline{r}+\delta\underline{r}))(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r})$$
$$+ \delta\underline{r}(\underline{p}\cdot(\underline{r}+\delta\underline{r}))(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r})\Big\rangle \quad -(16)$$

$$= \frac{1}{r^2}\Big\langle \underline{r}(\underline{p}\cdot\underline{r})(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r})$$
$$+ \underline{r}(\underline{p}\cdot\delta\underline{r})(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r})$$
$$+ \delta\underline{r}(\underline{p}\cdot\underline{r})(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r})$$
$$+ \delta\underline{r}(\underline{p}\cdot\delta\underline{r})(2\underline{r}\cdot\delta\underline{r}+\delta\underline{r}\cdot\delta\underline{r})\Big\rangle \quad -(17)$$

There are eight terms as follows:

1) $\langle \underline{r}(\underline{p}\cdot\underline{r})2\underline{r}\cdot\delta\underline{r}\rangle = \underline{r}(\underline{p}\cdot\underline{r})2\underline{r}\cdot\langle\delta\underline{r}\rangle = \underline{0}$ $\quad -(18)$

2) $\langle \underline{r}(\underline{p}\cdot\underline{r})\delta\underline{r}\cdot\delta\underline{r}\rangle = \underline{r}(\underline{p}\cdot\underline{r})\langle\delta\underline{r}\cdot\delta\underline{r}\rangle$ $\quad -(19)$

3) $\langle \underline{r}(\underline{p}\cdot\delta\underline{r})(2\underline{r}\cdot\delta\underline{r})\rangle$

$$= \langle \underline{r}(p_x\delta X+p_Y\delta Y+p_Z\delta Z)(2(X\delta X+Y\delta Y+Z\delta Z))\rangle$$

$$= 2\underline{r}\langle p_X X\delta X^2+p_Y Y\delta Y^2+p_Z Z\delta Z^2\rangle$$

$$= \frac{2}{3}\underline{r}(\underline{p}\cdot\underline{r})\langle\delta\underline{r}\cdot\delta\underline{r}\rangle \quad -(20)$$

using eqs. (9) and (10).

4) $\underline{r}\langle(\underline{p}\cdot\delta\underline{r})\delta\underline{r}\cdot\delta\underline{r}\rangle$

$$= \underline{r}\langle(p_X\delta X+p_Y\delta Y+p_Z\delta Z)\delta\underline{r}\cdot\delta\underline{r}\rangle$$

$$= \underline{r} \langle p_x (\delta X)^3 + \cdots \rangle$$

$$= \underline{0} \quad -(21)$$

5) $\langle \underline{p} \cdot \underline{r}\, \delta \underline{r} (\delta \underline{r} \cdot \delta \underline{r}) \rangle = \underline{p} \cdot \underline{r} \langle \delta \underline{r} (\delta \underline{r} \cdot \delta \underline{r}) \rangle = \underline{0}$ $-(22)$

6) $2 \langle (\underline{p} \cdot \underline{r})\, \delta \underline{r}\, \underline{r} \cdot \delta \underline{r} \rangle$

$$= 2\, \underline{p} \cdot \underline{r} \langle \delta \underline{r} (X \delta X + Y \delta Y + Z \delta Z) \rangle$$

$$= \frac{1}{3} \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \underline{r} \quad -(23)$$

using eqs. (9) and (10)

7) $\langle \delta \underline{r} (\underline{p} \cdot \delta \underline{r})\, 2\, \underline{r} \cdot \delta \underline{r} \rangle$

$$= 2 \langle \delta \underline{r}\, \underline{p} \cdot (X\, \delta X^2\, \underline{i} + \cdots) \rangle \quad -(24)$$

$$= 2\, \underline{i}\, p_x X \langle \delta X^3 \rangle = \underline{0}$$

8) $\langle \delta \underline{r} (\underline{p} \cdot \delta \underline{r})\, \delta \underline{r} \cdot \delta \underline{r} \rangle$

$$= \langle \delta \underline{r} (p_x \delta X + p_y \delta Y + p_z \delta Z)(\delta X^2 + \delta Y^2 + \delta Z^2) \rangle$$

$$= \underline{i} \langle \delta X (p_x \delta X + p_y \delta Y + p_z \delta Z)(\delta X^2 + \delta Y^2 + \delta Z^2) \rangle$$

$$+ \cdots$$

$$= \underline{i}\, p_x \langle (\delta X)^4 \rangle + \cdots$$

$$= \frac{1}{9} \underline{p} \langle (\delta \underline{r} \cdot \delta \underline{r})(\delta \underline{r} \cdot \delta \underline{r}) \rangle \quad -(25)$$

Therefore adding these eight terms:

b) $$\underline{E}_1 = \frac{5}{6\pi\epsilon_0 r^5}\left(\frac{7}{3}\underline{r}(\underline{p}\cdot\underline{r})\langle\delta\underline{r}\cdot\delta\underline{r}\rangle + \frac{1}{9}\underline{p}\left\langle(\delta\underline{r}\cdot\delta\underline{r})^2\right\rangle\right) \quad -(26)$$

So the vacuum corrected electric dipole field is:

$$\begin{aligned}\underline{E} = {} & \frac{1}{4\pi\epsilon_0 r^3}\left(\frac{(\underline{r}\cdot\underline{p})\underline{r}}{r^2} - \underline{p}\right) + \frac{11}{24}\underline{p}\frac{\langle\delta\underline{r}\cdot\delta\underline{r}\rangle}{4\pi\epsilon_0 r^5} \\ & - \frac{35}{24\pi\epsilon_0 r^5}\underline{r}(\underline{p}\cdot\underline{r})\langle\delta\underline{r}\cdot\delta\underline{r}\rangle \\ & + \frac{1}{9}\underline{p}\left\langle(\delta\underline{r}\cdot\delta\underline{r})^2\right\rangle\end{aligned} \quad -(27)$$

to first order in x.

A very rich structure emerges.

In general these calculations must be carried out with computer algebra, in order to isolate terms in $\langle\delta\underline{r}\cdot\delta\underline{r}\rangle$ and higher orders thereof. Eq. (27) can be graphed and compared with the well known dipole field in the hypothetical absence of the vacuum, the first term on the RHS of eq. (27)