591(9): Checks a Method: The Universal Effective Eccentricity

In the classical limit:

$$H = \frac{1}{2} m v_N^2 - \frac{m M G}{r} \quad - (1)$$

where

$$v_N^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad - (2)$$

and

$$H = -\frac{m M G}{2a} \quad - (3)$$

so

$$\frac{m M G}{\alpha}\left(1 + \epsilon\cos\phi\right) = \frac{1}{2} m v_N^2 - H \quad - (4)$$

At the perihelion:

$$v_N^2 = \frac{MG}{\alpha}(1+\epsilon)^2, \quad - (5)$$

and the hamiltonian is:

$$H = -\frac{m M G}{2a} = -\frac{m M G}{2}\left(\frac{1-\epsilon^2}{\alpha}\right) - (6)$$

so

$$\frac{m M G}{\alpha}\left(1 + \epsilon\cos\phi\right) = \frac{m M G}{2\alpha}(1+\epsilon)^2 + \frac{m M G}{2\alpha}\left(1-\epsilon^2\right) \quad - (7)$$

i.e.

$$1 + \epsilon\cos\phi = \frac{1}{2}\left((1+\epsilon)^2 + 1 - \epsilon^2\right) \quad - (8)$$
$$= 1 + \epsilon$$

and

$$\cos\phi = 1, \quad \phi = 2\pi \checkmark\checkmark \quad - (9)$$

The relativistic hamiltonian is:

$$H = \gamma m c^2 - \frac{m M G}{r} \quad - (10)$$

so

$$H_0 = H - mc^2 = (\gamma - 1) mc^2 - \frac{m M G}{r} \quad - (11)$$

where

$$T = (\gamma - 1) mc^2 \quad - (12)$$

is the relativistic kinetic energy.

) So:

$$T = \left(\left(1 - \frac{v_N^2}{c^2} \right)^{-1/2} - 1 \right) mc^2 \quad -(13)$$

In the approximation:

$$\left(1 - \frac{v_N^2}{c^2} \right)^{-1/2} \sim 1 + \frac{1}{2} \frac{v_N^2}{c^2} \quad -(14)$$

$$T \rightarrow \frac{1}{2} m v_N^2 \quad -(15)$$

More accurately:

$$\left(1 - \frac{v_N^2}{c^2} \right)^{-1/2} \sim 1 + \frac{1}{2} \frac{v_N^2}{c^2} + \frac{3}{8} \frac{v_N^4}{c^4} + \frac{5}{16} \frac{v_N^6}{c^6} + \ldots \quad -(16)$$

Therefore the first term in this expansion leads to the Newtonian result (15). The other terms give relativistic corrections.

Now assume that the hamiltonian (11) gives the orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos \phi_1} \quad -(17)$$

where

$$\phi_1 = \phi + \Delta \phi \quad -(18)$$

It follows that

$$\frac{mMG}{r} = (\gamma - 1) mc^2 - H_0 = \frac{mMG}{\alpha} (1 + \epsilon \cos \phi_1) \quad -(19)$$

So

$$1 + \epsilon \cos \phi_1 = \frac{\alpha}{mMG} \left((\gamma - 1) mc^2 - H_0 \right) \quad -(20)$$

3)

So

$$\cos\phi_1 = \frac{\alpha c^2}{mG\epsilon}(\gamma - 1) - \frac{1}{\epsilon}\left(\frac{\alpha H_0}{mmG} - 1\right) \quad -(21)$$

At the perihelion:

$$v_N^2 = \frac{mG}{\alpha}(1+\epsilon)^2 \quad -(22)$$

The Newtonian limit of eq. (21) is obtained from

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \sim 1 + \frac{v^2}{c^2} \quad -(23)$$

and

$$H_0 = -\frac{mMG}{a} \quad -(24)$$

$$= -\frac{mMG(1-\epsilon^2)}{\alpha}$$

so

$$\cos\phi_1 = \frac{1}{2\epsilon}(1+\epsilon^2) + \frac{1}{\epsilon}\left(\frac{1-\epsilon^2}{2} - 1\right) \quad -(25)$$

$$= \frac{1}{2\epsilon}(1+\epsilon^2) - \frac{1}{2\epsilon}(1+\epsilon^2)$$

$$= 0$$

This means that in the Newtonian limit there is no precession, a self consistent result, Q.E.D.

Now use the binomial expansion (16) to find that

$$\cos\phi_1 = \frac{3}{8\epsilon}\left(\frac{mG}{\alpha c^2}\right)(1+\epsilon)^4 + \frac{5}{16\epsilon}\left(\frac{mG}{\alpha c^2}\right)^2(1+\epsilon)^6 + \ldots \quad -(26)$$

if it is assumed that eq. (24) is approximately true.

so $\cos \phi$ is no longer zero. It has advanced by

very small angle $\Delta \phi$ defined by:

$$\phi_1 = \phi + \Delta \phi \quad -(27)$$

so

$$\cos \phi_1 = \cos \Delta \phi \quad -(28)$$

at the perihelion

$$\phi = 2\pi \quad -(29)$$

So:

$$\Delta \phi = \cos^{-1}\left(\frac{3}{8\epsilon}\left(\frac{mG}{\alpha c^2}\right)(1+\epsilon)^4 + \frac{5}{16\epsilon}\left(\frac{mG}{\alpha c^2}\right)^2 (1+\epsilon)^6 + \cdots \right) \quad -(30)$$

The experimental result is:

$$\Delta \phi = \cos^{-1}\left(\frac{6\pi mG}{\alpha c^2} \right) \quad -(31)$$

so exact agreement is obtained when:

$$\frac{3}{8\epsilon}(1+\epsilon)^4 + \frac{5}{16\epsilon}(1+\epsilon)^6 \left(\frac{mG}{\alpha c^2}\right) + \cdots = 6\pi \quad -(32)$$

The first term on the LHS dominates, so

$$\boxed{\frac{3}{8\epsilon}(1+\epsilon)^4 = 6\pi} \quad -(33)$$

This is an effective eccentricity because the classical eq. (24) has been used. To be rigorously self consistent the relativistic hamiltonian must be used. The effective eccentricity is the same for all precessing orbits of this type:

$$r = \frac{\alpha}{1 + \epsilon_0 \cos(\phi + \Delta \phi)} \quad -(34)$$

5) Therefore
$$\boxed{\epsilon_0 = 0.0217} \quad -(35)$$
to obtain an orbit of type (34) with precession:
$$\Delta\phi = \frac{6\pi m G}{\alpha c^2} \quad -(36)$$
per orbit.

From eq. (21):
$$\Delta\phi = \cos^{-1}\left[\frac{\alpha c^2}{mG\epsilon}(\gamma - 1) - \frac{1}{\epsilon}\left(\frac{\alpha H_0}{m m G} - 1\right)\right] \quad -(37)$$
and using the experimental α and ϵ for each orbit, adjust H_0 to obtain:
$$\frac{\alpha c^2}{mG\epsilon}(\gamma - 1) - \frac{1}{\epsilon}\left(\frac{\alpha H_0}{m M G} - 1\right) = \frac{6\pi m G}{\alpha c^2} \quad -(38)$$
where
$$\gamma = \left(1 - \frac{mG}{\alpha c^2}(1+\epsilon)^2\right)^{-1/2} \quad -(39)$$