371(1): Spherical Polar Coordinates and the Precession of the Perihelia of Planets.

Consider a mass m orbiting a mass M in three dimensions. The relevant lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) \right) - U \quad -(1)$$

where

$$U = -\frac{mMG}{r} \quad -(2)$$

Here G is Newton's constant and r the distance between m and M. There are three Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) \quad -(3)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) \quad -(4)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \quad -(5)$$

which can be solved simultaneously with Maxima to give $r(t)$, $\theta(t)$, $\phi(t)$, $\dot{r}(t)$, $\dot{\theta}(t)$ and $\dot{\phi}(t)$.

As in UFT270 define:

$$\dot{\beta} = \dot{\theta} + \dot{\phi} \sin^2\theta \quad -(6)$$

so the lagrangian becomes:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right) + \frac{mMG}{r} \quad -(7)$$

Eq. (7) gives the orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(8)$$

[wh]ere α is the half right latitude and ϵ the [e]ccentricity. Eq. (8) emerges from the Euler Lagrange [e]quation:

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) \quad -(9)$$

Eq. (9) gives

$$\frac{dL}{dt} = 0 \quad -(10)$$

where

$$L = mr^2\dot{\beta} \quad -(11)$$

is a constant angular momentum of the motion. So

$$\dot{\beta} = \frac{L}{mr^2} = \frac{L}{m\alpha^2}\left(1+\cos\beta\right)^2 - (12)$$

i.e.

$$\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta = \frac{L}{m\alpha^2}\left(1+\cos\beta\right)^2 - (13)$$

Eqs. (3), (4), (5), (9) and (13) can be [s]olved simultaneously with Maxima to give [β](t), $\dot{\beta}$(t) and to express β as a function [o]f the angle θ and ϕ of the spherical polar coordinates. The precession of the perihelion in a planar orbit is a precession in the angle ϕ. This is one of the various precessions present.