69(1): Motion of a Gyroscope in a Gravitational Field.

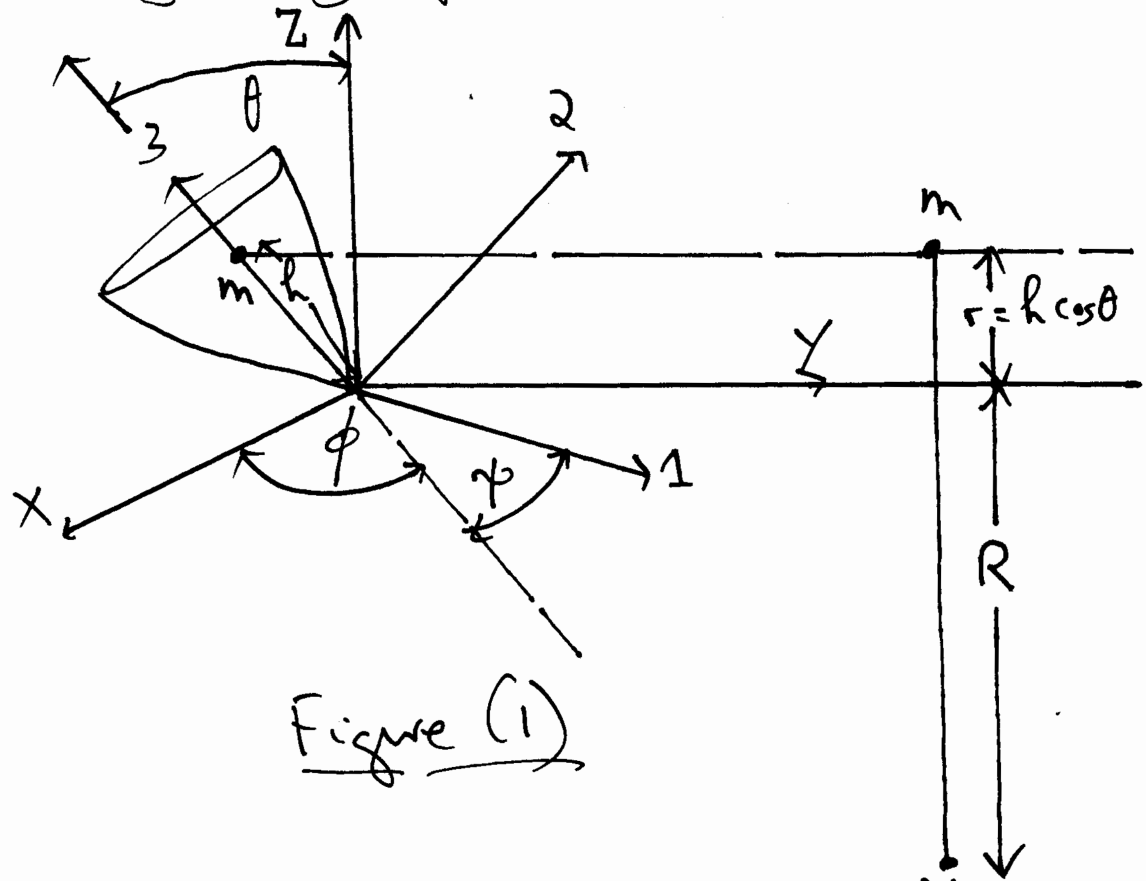

Figure (1)

In Fig (1) the point of the gyro is allowed to move in the gravitational field of the earth of mass M and radius R.

The Lagrangian is therefore made up of translational and rotational motions:

$$\mathcal{L} = T_{trans} + T_{rot} - U \quad -(1)$$

where T_{trans} is the translational kinetic energy and T_{rot} is the rotational kinetic energy. The potential energy is

$$U = mg\,(h\cos\theta + R) \quad -(2)$$

where R is the radius of the earth and

$$r = h\cos\theta. \quad -(3)$$

Here h is the constant distance from the origin to the centre of mass of the gyroscope along the 3 axis of frame (1, 2, 3). This is the frame of the principal moments of inertia of the gyroscope:

$$I_{12} = I_1 = I_2 \text{ and } I_3. \quad -(4)$$

The translational kinetic energy is therefore:

$$T_{trans} = \frac{1}{2}\frac{mM}{m+M}\underline{v}\cdot\underline{v} \quad -(5)$$

where:

$$\underline{v} = \frac{d}{dt}(\underline{r}+\underline{R}) = \dot{\underline{r}} \quad -(6)$$

because:

$$\frac{d\underline{R}}{dt} = \underline{0}. \quad -(7)$$

Therefore:

$$\boxed{T_{trans} = \frac{1}{2}m\dot{r}^2} \quad -(8)$$

because

$$M \gg m. \quad -(9)$$

The rotational kinetic energy is as defined in UFT368:

$$T_{rot} = \frac{1}{2}I_{12}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 \quad -(10)$$

where the Euler angles θ, ϕ and ψ are defined in Fig. (1).

3) The Lagrangian (1) is therefore:

$$\mathcal{L} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 - mg(r+R) \quad -(11)$$

in which

$$r = h\cos\theta \quad -(12)$$

There are four Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) \quad -(13)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) \quad -(14)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \quad -(15)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) \quad -(16)$$

Eqs. (15) and (16) give the same results as in UFT 348:

$$\dot{\phi} = \frac{L_\phi - L_\psi \cos\theta}{I_{12} \sin^2\theta} \quad -(17)$$

$$\dot{\psi} = \frac{1}{I_3} \left(L_\psi - I_3 \dot{\phi} \cos\theta \right) \quad -(18)$$

Eq. (14) gives:

$$\ddot{\theta} = \frac{\sin\theta}{I_{12}} \left(\dot{\phi}^2 \cos\theta \left(I_{12} - I_3 \right) - I_3 \dot{\phi} \dot{\psi} + mgh \right) \quad -(19)$$

because r in the potential energy term of eq. (11) is defined by eq. (12). Eq. (19) is also the same as in UFT 368.

However, there is another equation (13) which gives:

$$-mg = m\ddot{r} \quad -(20)$$

i.e.

$$\ddot{r} = -g = \frac{MG}{R^2} \quad -(21)$$

This is the correct result for the motion of r if it assumed that R is a constant. This assumes that the point of the gyro does not move with respect to the centre of the earth. More generally R depends on time. Defining:

$$\underline{R}_1 = \underline{r} + \underline{R} \quad -(22)$$

the kinetic energy is:

$$T = \frac{1}{2} m \underline{R}_1 \cdot \underline{R}_1 = \frac{1}{2} m R_1^2 \quad -(23)$$

and the potential energy is:

$$U = mgR_1 \quad -(24)$$

and

$$\frac{\partial \mathcal{L}}{\partial R_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{R}_1}\right) \quad -(25)$$

Eq. (25) means that the point of the gyroscope can move up or down.

5) The lagrangian in Eq. (25) is

$$\mathcal{L} = \frac{1}{2} m \dot{R}_1^2 + \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 - \frac{mMG}{R_1} \quad -(26)$$

in which

$$R_1 = r + R = h\cos\theta + R \quad -(27)$$

Eqs. (25) and (26) give:

$$\ddot{R}_1 = \frac{MG}{R_1^2} \quad -(28)$$

i.e

$$\frac{d^2}{dt^2}\left(h\cos\theta(t) + R(t) \right) = \frac{MG}{(h\cos\theta + R)^2} \quad -(29)$$

i.e

$$\ddot{R} + h\frac{d^2}{dt^2}\left(\cos\theta(t)\right) = \frac{MG}{(h\cos\theta + R)^2} \quad -(30)$$

Finally use:

$$\frac{d^2 \cos\theta(t)}{dt^2} = -\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta \right) \quad -(31)$$

so

$$\boxed{\ddot{R} - h\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta \right) = \frac{MG}{(h\cos\theta + R)^2}} \quad -(32)$$

Therefore eqs. (17) to (19) and (32) must be solved simultaneously to obtain $R(t)$.