368(8): Analytical Mechanics of Torque in a Gyroscope

In general the torque is:

$$Tq = \frac{dL}{dt} = \left(\frac{d\underline{L}}{dt} + \underline{\omega} \times \underline{L} \right)_{123} \quad -(1)$$

where (1, 2, 3) is the frame of the principal moments of inertia of the gyroscope. Here $\underline{L}$ is the angular momentum. The Euler equations follow:

$$Tq_1 = I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \quad -(2)$$
$$Tq_2 = I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \quad -(3)$$
$$Tq_3 = I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 \quad -(4)$$

In terms of the Euler angles θ, ϕ, ψ:

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \quad -(5)$$
$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \quad -(6)$$
$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi} \quad -(7)$$

It follows that:

$$\dot{\omega}_1 = \frac{d}{dt}\left(\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \right) \quad -(8)$$
$$\dot{\omega}_2 = \frac{d}{dt}\left(\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \right) \quad -(9)$$
$$\dot{\omega}_3 = \frac{d}{dt}\left(\dot{\phi} \cos\theta + \dot{\psi} \right) \quad -(10)$$

2) Note carefully that:

$$\theta = \theta(t), \quad \phi = \phi(t), \quad \psi = \psi(t) \quad -(11)$$

so if for example:

$$y = \cos\theta \quad -(12)$$

then

$$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\sin\theta \quad -(13)$$

and so on.

So:

$$\dot{\omega}_3 = \ddot{\phi}\cos\theta + \dot{\phi}\frac{d\cos\theta}{dt} + \ddot{\psi}$$

$$= \ddot{\phi}\left(\cos\theta - \dot{\theta}\sin\theta\right) + \ddot{\psi} \quad -(14)$$

Note that:

$$\frac{d}{dt}\left(\dot{\phi}\sin\theta\sin\psi\right) = \ddot{\phi}\sin\theta\sin\psi + \dot{\phi}\frac{d}{dt}\left(\sin\theta\sin\psi\right)$$

$$= \ddot{\phi}\sin\theta\sin\psi + \dot{\phi}\left(\sin\psi\frac{d}{dt}\sin\theta + \sin\theta\frac{d}{dt}\sin\psi\right)$$

$$= \ddot{\phi}\sin\theta\sin\psi + \dot{\phi}\left(\dot{\theta}\sin\psi\cos\theta + \dot{\psi}\sin\theta\cos\psi\right) \quad -(15)$$

It follows that:

$$\dot{\omega}_1 = \ddot{\phi}\sin\theta\sin\psi + \dot{\phi}\left(\dot{\theta}\sin\psi\cos\theta + \dot{\psi}\sin\theta\cos\psi\right) + \ddot{\theta}\cos\psi - \dot{\theta}\dot{\psi}\sin\psi \quad -(16)$$

3)

and

$$\dot{\omega}_2 = \ddot{\phi}\sin\theta\sin\psi + \dot{\phi}\left(\dot{\theta}\sin\psi\cos\theta + \dot{\psi}\sin\theta\cos\psi\right)$$
$$-\ddot{\theta}\sin\psi + \dot{\theta}\dot{\psi}\cos\psi \quad -(17)$$

From the Euler Lagrange analysis of previous work:

$$\ddot{\theta} = \frac{\sin\theta}{I_{12}}\left(\dot{\phi}^2\cos\theta\left(I_{12} - I_3\right) - I_3\dot{\phi}\dot{\psi} + mgh\right) \quad -(18)$$

$$\dot{\phi} = \frac{L_\phi - L_\psi\cos\theta}{I_{12}\sin^2\theta} \quad -(19)$$

$$\dot{\psi} = \frac{1}{I_3}\left(L_\psi - I_3\dot{\phi}\cos\theta\right) \quad -(20)$$

where

$$I_{12} = I_1 = I_2 \quad -(21)$$

Therefore the torque components in the frame (1, 2, 3) of the principal moments of inertia of the gyroscope (symmetric top) can be found from the Euler angles, θ, ϕ and ψ and from $\ddot{\theta}$, $\ddot{\phi}$, $\ddot{\psi}$, $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$. This can be done by computer algebra.

†) The complete moving frame torque is:

$$\underline{Tq} = Tq_1\,\underline{e}_1 + Tq_2\,\underline{e}_2 + Tq_3\,\underline{e}_3 \quad -(23)$$

In the laboratory frame (X, Y, Z) the torque is:

$$\underline{Tq} = \underline{r} \times \underline{F} = Tq_X\,\underline{i} + Tq_Y\,\underline{j} + Tq_Z\,\underline{k} \quad (24)$$

where the force of gravity at the centre of mass of the gyroscope is:

$$\underline{F} = mg\,\underline{k} \quad -(25)$$

In general:

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(26)$$

so

$$\underline{Tq} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ X & Y & Z \\ 0 & 0 & mg \end{vmatrix} \quad -(27)$$

$$= mg\left(Y\underline{i} - X\underline{j}\right)$$

The origin of (X, Y, Z) and $(1, 2, 3)$ are the same,

so:

$$Tq_X^2 + Tq_Y^2 + Tq_Z^2 \quad -(28)$$

$$= Tq_1^2 + Tq_2^2 + Tq_3^2$$

$$= m^2 g^2 \left(Y^2 + X^2\right)$$

In the symmetric top with one point fixed:

$$X^2 + Y^2 = h^2 \quad -(29)$$

where h is the constant distance from the origin to the centre of mass of the gyroscope. So:

$$T_{a_{V_1}}^2 + T_{a_{V_2}}^2 + T_{a_{V_3}}^2 = m^2 g^2 h^2 \quad -(30)$$

The gravitational force on the centre of mass of the gyroscope is, in frame (X, Y, Z):

$$\underline{F} = mg\,\underline{k} \quad -(31)$$

So from eqs. (30) and (31):

$$\boxed{F^2 = \frac{1}{h^2}\left(T_{a_{V_1}}^2 + T_{a_{V_2}}^2 + T_{a_{V_3}}^2\right)} \quad -(32)$$

and

$$F = \pm\frac{1}{h}\left(T_{a_{V_1}}^2 + T_{a_{V_2}}^2 + T_{a_{V_3}}^2\right)^{1/2} \quad -(33)$$

The downward force of gravitation is:

$$\underline{F} = mg\,\underline{k} = -\frac{mMG}{R^2}\,\underline{k} \quad -(34)$$

so

$$g = -\frac{MG}{R^2} \quad -(35)$$

From the foregoing analysis:

$$g^2 = \frac{1}{m^2 h^2}\left(T_{a_{V_1}}^2 + T_{a_{V_2}}^2 + T_{a_{V_3}}^2\right) \quad -(36)$$

6) Taking the positive solution:

$$g_+ = \frac{1}{mh}\left(T_{q_{V_1}}^2 + T_{q_{V_2}}^2 + T_{q_{V_3}}^2\right) \quad -(37)$$

so

$$\underline{F}(\text{gyro}) = g_+ \, \underline{k} \quad -(38)$$

and

$$\underline{F}(\text{grav}) = -\frac{mG}{R^2}\,\underline{k} \quad -(39)$$

It follows that:

$$\boxed{\underline{F}(\text{gyro}) = -\underline{F}(\text{grav})} \quad -(40)$$

and the gyro appears weightless in certain configurations