368(6): Final Version of Note 368(5).

Using the Lagrangian:

$$\mathcal{L} = \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 - mgh\cos\theta \quad -(1)$$

and the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) \quad -(2)$$

it is found that:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I_{12}\dot{\theta} \quad -(3)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = \sin\theta \left(I_{12} \dot{\phi}^2 \cos\theta - I_3 \dot{\phi}^2 \cos\theta - I_3 \dot{\phi} \dot{\psi} + mgh \right) \quad -(4)$$

so:

$$I_{12} \frac{d^2\theta}{dt^2} = \sin\theta \left[\dot{\phi}^2 \cos\theta \left(I_{12} - I_3 \right) - I_3 \dot{\phi} \dot{\psi} + mgh \right] \quad -(5)$$

where:

$$\dot{\phi} = \frac{L_\phi - L_\psi \cos\theta}{I_{12} \sin^2\theta} \quad -(6)$$

and

$$\dot{\psi} = \frac{1}{I_3}\left(L_\psi - I_3 \dot{\phi} \cos\theta \right) \quad -(7)$$

2) Here:

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} = \frac{L_\phi}{I_3} = \text{constant} \quad -(8)$$

and

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \quad -(9)$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \quad -(10)$$

Therefore $\frac{d^2\theta}{dt^2}$, $\frac{d\phi}{dt}$ and $\frac{d\psi}{dt}$ can be graphed against the angle θ using eqns. (5), (6) and (7). An animation can be made by making $\theta(t)$ time dependent. Eq. (5) can be solved for $\theta(t)$ numerically and θ plotted as a function of time.

In these equations the point of the gyroscope does not move by definition, it is defined as the origin of the (X, Y, Z) and $(1, 2, 3)$ frames. In order to consider the motion of the centre of mass of the gyroscope with respect to the stationary frame, the Lagrangian and Hamiltonian must be defined in spherical polar coordinates, (r, ϕ, θ), so:

$$H = \frac{1}{2}mv^2 + U \quad -(11)$$

$$\mathcal{L} = \frac{1}{2}mv^2 - U \quad -(12)$$

This will be the subject of the next note.