8(5): Complete Dynamics of the Gyroscope: a Summary

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) - mgh\cos\theta \quad -(1)$$

here

$$I_1 = I_2 = I_{12} \quad -(2)$$

and

$$\dot{\beta} = \omega_1^2 + \omega_2^2 = \dot{\phi}^2\sin^2\theta + \dot{\theta}^2 \quad -(3)$$

$$\dot{\alpha} = \omega_3^2 = (\dot{\phi}\cos\theta + \dot{\psi})^2 \quad -(4)$$

For free rotation:

$$\mathcal{L} = \frac{1}{2}\left(I_{12}\dot{\beta}^2 + I_3\dot{\alpha}^2\right). \quad -(5)$$

The Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \quad -(6)$$

and

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\beta}} \quad -(7)$$

imply:

$$\frac{d}{dt}\left(I_{12}\dot{\beta}\right) = \frac{d}{dt}\left(I_3\dot{\alpha}\right) = 0 \quad -(8)$$

so

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = \text{constant} \quad -(9)$$

where T is the kinetic energy defined for free rotation by

$$H = T \quad -(10)$$

In the presence of a potential:

$$\mathcal{L} = \frac{1}{2}\left(I_{12}\dot{\beta}^2 + I_3\dot{\alpha}^2\right) - mgl\cos\theta \quad -(11)$$

and eq. (7) gives:

$$\frac{d}{dt}\left(I_{12}\dot{\beta}\right) = mgh\left(\frac{\partial\theta}{\partial\beta}\right)\sin\theta \quad -(12)$$

using:

$$\frac{\partial\mathcal{L}}{\partial\beta} = \frac{\partial\mathcal{L}}{\partial\theta}\frac{\partial\theta}{\partial\beta} = mgh\left(\frac{\partial\theta}{\partial\beta}\right)\sin\theta \quad -(13)$$

This is a development of Eq. (10) of the previous note. There are also the other Euler Lagrange equations in the Euler angles θ, ϕ and ψ.

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \quad -(14)$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} \quad -(15)$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \quad -(16)$$

These give:

$$L_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \left(I_{12}\sin^2\theta + I_3\cos^2\theta\right)\dot{\phi} + I_3\dot{\psi}\cos\theta$$
$$= \text{constant} \quad -(17)$$

$$L_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)$$
$$= \text{constant} \quad -(18)$$

and

3)

$$I_{12}\ddot{\theta} = I_3\dot{\phi}^2\sin\theta\cos\theta \quad -(19)$$

It follows that:

$$\frac{d\phi}{dt} = \dot{\phi} = \frac{L_\phi - L_\psi\cos\theta}{I_{12}\sin^2\theta} \quad -(20)$$

and

$$\frac{d\psi}{dt} = \dot{\psi} = \frac{(L_\phi - L_\psi\cos\theta)\cos\theta}{I_{12}\sin^2\theta} \quad -(21)$$

So

$$I_{12}\frac{d^2\theta}{dt^2} = I_3\left(\frac{L_\phi - L_\psi\cos\theta}{I_{12}\sin^2\theta}\right)^2\sin\theta\cos\theta \quad -(22)$$

It also follows that:

$$L_\psi = I_3\omega_3 = \text{constant} \quad -(23)$$

where

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \quad -(24)$$

So

$$\left(\frac{L_\phi - L_\psi\cos\theta}{I_{12}\sin^2\theta}\right)\cos\theta + \left(\frac{L_\phi - L_\psi\cos\theta}{I_{12}\sin^2\theta}\right)\cos\theta$$

$$= 2\left(\frac{L_\phi - L_\psi\cos\theta}{I_{12}\sin^2\theta}\right)\cos\theta = \text{constant}$$

$$= L_\psi / I_3 \quad -(25)$$

) So

$$\frac{L_\phi - L_\psi \cos\theta}{I_{12}\sin^2\theta} = \frac{L_\psi}{2 I_3 \cos\theta} \quad -(26)$$

Fm eqs. (22) and (26):

$$\boxed{I_{12}\frac{d^2\theta}{dt^2} = \left(\frac{L_\psi}{2I_3}\right)^2 \tan\theta} \quad -(27)$$

Fm eqs. (20) and (26):

$$\boxed{\frac{d\phi}{dt} = \frac{L_\psi}{2I_3\cos\theta}} \quad -(28)$$

Fm eqs. (21) and (26):

$$\boxed{\frac{d\psi}{dt} = \frac{L_\psi}{2I_3}} \quad -(29)$$

o

$$\psi = \frac{L_\psi}{2I_3} t \quad -(30)$$

d increases with time. Eq. (27) can be solved numerically θ(t), and $d\phi/dt$ can found knowing θ(t).

Fig (1)