68(4): Pure Rotational Motion with One End Fixed

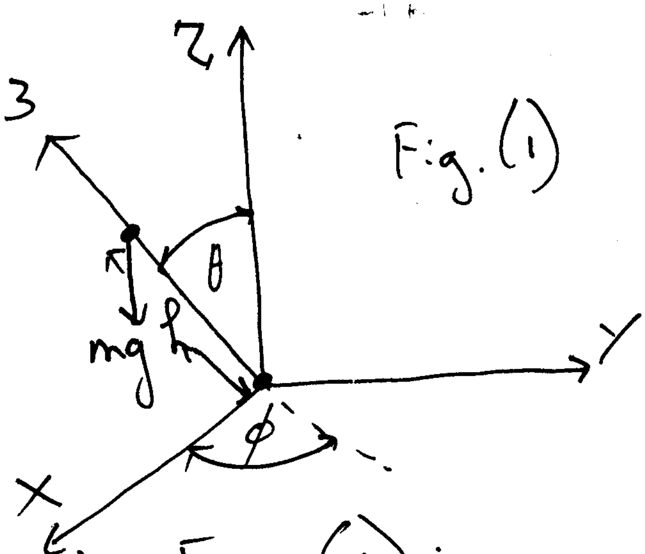

Fig. (1)

With reference to Fig (1):

$$X = h \sin\theta \cos\phi \qquad -(1)$$
$$Y = h \sin\theta \sin\phi$$
$$Z = h \cos\theta$$

where h is a constant.

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m h^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - mgh\cos\theta$$
$$= \frac{1}{2} m h^2 \dot{\beta}^2 - mgh\cos\theta \qquad -(2)$$
$$= \frac{1}{2} I_{12} \left(\omega_1^2 + \omega_2^2\right) - mgh\cos\theta, \qquad -(3)$$

$$I_{12} = I_1 = I_2$$

The origin of (X, Y, Z) and $(1, 2, 3)$ is the same.

The complete Lagrangian is:

$$\mathcal{L} = \frac{1}{2} I_{12}\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2 - mgh\cos\theta \qquad -(4)$$

) which is that of a symmetric top:

$$I_{12} = I_1 = I_2 \text{ and } I_3 \quad -(5)$$

Therefore the complete lagrangian is:

$$\mathcal{L} = \frac{1}{2} m h^2 \dot{\beta}^2 + \frac{1}{2} I_3 \omega_3^2 - mgh\cos\theta \quad -(6)$$

In terms of Euler angles:

$$\omega_3^2 = (\dot{\phi}\cos\theta + \dot{\psi})^2 \quad -(7)$$

Consider first the lagrangian (2) with the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\beta}} \quad -(8)$$

It follows that:

$$\frac{dL_\beta}{dt} = 0 \quad -(9)$$

where

$$L_\beta = m h^2 \dot{\beta} \quad -(10)$$

is a constant angular momentum. Therefore:

$$\dot{\beta} = (\omega_1^2 + \omega_2^2)^{1/2} = \frac{L}{mh^2} \quad -(11)$$

$$= \text{constant}$$

The hamiltonian H is also a constant of motion, and:

$$H = \frac{1}{2}\frac{L^2}{ml^2} + mgl\cos\theta = \text{constant} \quad -(12)$$

so:

$$\cos\theta = \frac{1}{mgl}\left(H - \frac{1}{2}\frac{L}{ml^2}\right) = \text{constant} \quad -(13)$$

Now use the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(14)$$

with the lagrangian (2). It follows that:

$$L_Z = ml^2 \dot{\phi} \sin^2\theta \quad -(15)$$

is a constant of motion:

$$\frac{dL_Z}{dt} = 0 \quad -(16)$$

From eqs. (13) and (15) it follows that:

$$\dot{\phi} = \frac{L_Z}{ml^2 \sin^2\theta} = \text{constant} \quad -(17)$$

From eq. (11), $\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta = \text{constant} \quad -(18)$

so

$$\dot{\theta} = \text{constant.} \quad -(19)$$

From eq. (13) it follows that

$$\dot{\theta} = 0 \quad -(20)$$

Therefore the axis 3 precesses around the Z axis with

$$\frac{d\phi}{dt} = \frac{L_Z}{m l^2 \sin^2\theta} = \text{constant} \quad -(21)$$

The complete lagrangian is:

$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\beta}^2 + \frac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2 - m g l \cos\theta \quad -(22)$$

and there is another Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \quad -(23)$$

in addition to eqns. (8) and (14).

Eq. (23) gives:

$$L_3 = I_3 \omega_3 = \text{constant} \quad -(24)$$

where:

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta \quad -(25)$$

so

$$\boxed{\omega_3 = \frac{L_3}{I_3} = \text{constant}} \quad -(26)$$

From eqs. (22) and (8):

$$\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta = \frac{L^2}{m^2 l^4} = \text{constant} \quad -(27)$$

) and :

$$\dot{\psi} + \dot{\phi}\cos\theta = \frac{L_{\psi}}{I_3} = \text{constant} \quad -(28)$$

From eqs. (14) and (22):

$$L_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \left(I_{12}\sin^2\theta + I_3\cos^2\theta\right)\dot{\phi} + I_3\dot{\psi}\cos\theta$$
$$= \text{constant} \quad -(29)$$

The complete solution of the problem is given by eqs. (27), (28) and (29):

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta = \frac{L_{\psi}}{I_3} = \text{constant} \quad -(30)$$

$$\omega_1^2 + \omega_2^2 = \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta = \left(\frac{L_\beta}{m h^2}\right)^2 = \text{constant} \quad -(31)$$

and

$$\left(I_{12}\sin^2\theta + I_3\cos^2\theta\right)\dot{\phi} + I_3\dot{\psi}\cos\theta = L_\phi = \text{constant} \quad -(32)$$

Eqs. (30) and (32) are given by Marion and Thornton, but Eq. (31) is new.

From eqs. (30) and (32):

$$\frac{d\phi}{dt} = \frac{L_\phi - L_{\psi}\cos\theta}{I_{12}\sin^2\theta} \quad -(33)$$

and :

$$\frac{d\psi}{dt} = \frac{(L_\phi - L_\psi \cos\theta)\cos\theta}{I_{12}\sin^2\theta} \quad - (34)$$

Finally, $d\theta/dt$ can be found from eqs. (31) and (33):

$$\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{L_\beta}{m l^2}\right)^2 - \left(\frac{d\phi}{dt}\right)^2 \sin^2\theta$$

$$= \left(\frac{L_\beta}{m l^2}\right)^2 - \left(\frac{L_\phi - L_\psi \cos\theta}{I_{12}\sin^2\theta}\right)^2 \sin^2\theta$$

$$= \left(\frac{L_\beta}{m l^2}\right)^2 - \left(\frac{L_\phi - L_\psi \cos\theta}{I_{12}}\right)^2 \quad - (35)$$

The motion of the gyro is completely determined by eqs. (33), (34) and (35).