368(3): Expression for $Z(r)$

From eq. (33) of the previous note:

$$g\frac{\partial Z(r)}{\partial r} = \frac{L^2}{m^2 r^3} - \ddot{r} \quad -(1)$$

where

$$\dot{r} = \left(\frac{I_3}{m}\right)^{1/2} \omega_3 \quad -(2)$$

$$= \frac{1}{m I_3^{1/2}} I_3 \omega_3$$

So

$$\ddot{r} = \left(\frac{I_3}{m}\right)^{1/2} \frac{d\omega_3}{dt} \quad -(3)$$

and

$$g\frac{\partial Z(r)}{\partial r} = \frac{L^2}{m^2 r^3} - \left(\frac{I_3}{m}\right)^{1/2} \frac{d\omega_3}{dt} \quad -(4)$$

So:

$$g Z(r) = \frac{L^2}{m^2} \int \frac{dr}{r^3} - \left(\frac{I_3}{m}\right)^{1/2} \int \frac{d\omega_3}{dt} dr \quad -(5)$$

$$= -\frac{L^2}{2m^2 r^2} - \left(\frac{I_3}{m}\right)^{1/2} \int \frac{d\omega_3}{dt} dr$$

Now use:

$$\frac{d\omega_3}{dt} = \frac{d\omega_3}{dr}\frac{dr}{dt} = \left(\frac{I_3}{m}\right)^{1/2} \omega_3 \frac{d\omega_3}{dr} \quad -(6)$$

using:

$$\dot{r} = \frac{dr}{dt} = \left(\frac{I_3}{m}\right)^{1/2} \omega_3 \quad - (7)$$

Therefore:

$$mg\,Z(r) = -\frac{L^2}{2mr^2} - I_3 \int \omega_3 \, d\omega_3$$

$$= -\frac{L^2}{2mr^2} - \frac{I_3}{2}\omega_3^2 \quad - (8)$$

As in section 10.10 of Marion and Thornton:

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \quad - (9)$$

and

$$I_3 \omega_3 = A \quad - (10)$$

where A is a constant of motion. Therefore:

$$\boxed{mg\,Z(r) = -\frac{L^2}{2mr^2} - \frac{A^2}{2I_3}} \quad - (11)$$

It is seen that $Z(r)$ is always negative.

Eq. (11) is an inverse square law of attraction between m and M, where m is the mass of the gyro and M the mass of the earth. In eq. (11):

$$L^2 = m^2 r^4 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) = m^2 r^4 \left(\omega_1^2 + \omega_2^2\right)$$

$$= \text{constant of motion} \quad - (12)$$

3) The Lagrangian of the system is: —(13)

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)\right) - mgh\cos\theta$$

So using the Euler Lagrange Equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(14)$$

it is found that:

$$L_z = mr^2 \dot{\phi} \sin^2\theta \quad -(15)$$

is a constant of motion. It follows that the motion of θ is defined by:

$$\frac{d\theta}{dt} = \frac{1}{mr^2}\left(L^2 - \frac{L_z^2}{\sin^2\theta}\right)^{1/2} \quad -(16)$$

and that the motion of ϕ is defined by:

$$\frac{d\phi}{dt} = \frac{L_z}{mr^2\sin^2\theta} \quad -(17)$$

Using these equations $\tau_i(r)$ can be expressed in different ways and graphed.

Lift can be included by adding an external torque or force that results in a positive $\tau(r)$.