318(2): The Lagrangian of a Gyroscope in Spherical Polar Coordinates

Consider the gyroscope to be a symmetric top with one point fixed, with mass m with reference to Fig (1):

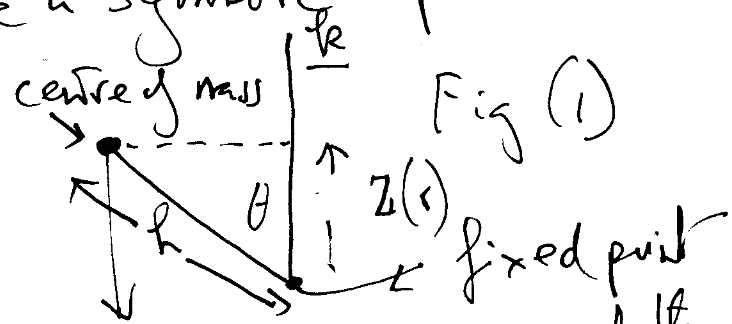

Fig (1)

$$Z = h\cos\theta \quad -(1)$$

so the potential energy of attraction between m and the mass of the earth M is

$$U = mgh\cos\theta \quad -(2)$$

In spherical polar coordinates the linear velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(3)$$

and the angular momentum is:

$$\underline{L} = m\,\underline{r}\times\underline{v} = mr^2\left(\dot{\theta}\,\underline{e}_\phi - \dot{\phi}\sin\theta\,\underline{e}_\theta\right) - (4)$$

Fm eq. (4):

$$L^2 = m^2r^4\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - (5)$$

is a constant of motion. The Z component of angular momentum is:

$$L_Z = mr^2\dot{\phi}\sin^2\theta \quad - (6)$$

The Lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2}mv^2 - U \quad -(7)$$

2) $$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) \right) - mgl\cos\theta \quad (8)$$

As in 4FT270, define:

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad (9)$$

so $$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right) - mgl\cos\theta \quad (10)$$

In terms of Euler angles (Marion and Thornton Section 10.10), the same Lagrangian is:

$$\mathcal{L} = \frac{1}{2} I_{12} \dot{\beta}^2 + \frac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2 - mgl\cos\theta \quad (11)$$

so $$I_{12} = mr^2 \quad (12)$$

and $$I_3 \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2 = m\dot{r}^2 \quad (13)$$

Also: $$\dot{\beta}^2 = \omega_1^2 + \omega_2^2 = \dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \quad (14)$$

and $$\omega_3^2 = \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2 = \frac{m\dot{r}^2}{I_3} \quad (15)$$

so the Lagrangian is:

$$\mathcal{L} = \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} I_{12} \left(\omega_1^2 + \omega_2^2 \right) - mgl\cos\theta \quad (16)$$

and the kinetic energy is:

$$T = \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} I_{12} \left(\omega_1^2 + \omega_2^2 \right) \quad (17)$$

3) The two Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(18)$$

and

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\beta}} \quad -(19)$$

Eqs. (10) and (18) give:

$$m\ddot{r} = mr\dot{\beta}^2 - mg\frac{\partial Z(r)}{\partial r} \quad -(20)$$

using

$$r = h \quad -(21)$$

in eq. (10).

Eqs (10) and (19) give:

$$\frac{\partial \mathcal{L}}{\partial \beta} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{\beta}} = m\dot{\beta}r^2 \quad -(22)$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) = 0 \quad -(23)$$

The angular momentum:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\beta}} = m\dot{\beta}r^2 \quad -(24)$$

is a constant of motion:

$$\frac{dL}{dt} = 0 \quad -(25)$$

4) Here:

$$\dot{\beta} = (\omega_1^2 + \omega_2^2)^{1/2} = (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)^{1/2} \quad - (\cancel{18}\ 26)$$

$$= \frac{L}{mr^2}$$

Fm eqs. (20) and (26):

$$\boxed{m\ddot{r} = \frac{L^2}{mr^3} - mg\frac{\partial Z(r)}{\partial r}} \quad - (27)$$

Fm eq. (15):

$$\boxed{m\dot{r}^2 = I_3\,\omega_3^2} \quad - (28)$$

Fm eq. (26): $$\boxed{\omega_1^2 + \omega_2^2 = \frac{L^2}{mr^4}} \quad - (29)$$

The Hamiltonian:

$$H = \frac{1}{2} I_3\,\omega_3^2 + \frac{1}{2} I_{12}(\omega_1^2 + \omega_2^2) + mg\,Z(r) \quad - (30)$$

is also a constant of motion:

$$\frac{dH}{dt} = 0 \quad - (31)$$

These are the equations of motion of the gyro developed as a symmetric top with one point fixed.

From eqs. (28), (29) and (30):

$$H = \frac{1}{2} m \dot{r}^2 + \frac{I_{12} L^2}{m r^4} + mg Z_1(r) \quad -(32)$$

and from eq. (27):

$$m \ddot{r} = \frac{L^2}{m r^3} - mg \frac{d Z_1(r)}{dr} \quad -(33)$$

with

$$\frac{dH}{dt} = 0 \quad -(34)$$

These are equations for the position $Z_1(r)$ of the gyro, the height of the centre of mass above the fixed point of the gyro.
