368(1): Constants of Motion of the Gyroscope

From UFT270 the angular momentum of the gyroscope is:

$$\underline{L} = mr^2 \left(\dot{\theta} \underline{e}_\phi - \dot{\phi} \sin\theta \, \underline{e}_\theta \right) \quad -(1)$$

in which

$$L_Z = mr^2 \dot{\phi} \sin^2\theta \quad -(2)$$

Also:

$$\dot{\phi} = \frac{L_Z}{mr^2 \sin^2\theta}, \quad \dot{\theta} = \frac{1}{mr^2} \left(L^2 - \frac{L_Z^2}{\sin^2\theta} \right)^{1/2} \quad -(3)$$

From eq. (20) of Note 367(8):

$$g = a_r \cos\theta - a_\theta \sin\theta. \quad -(4)$$

Here:

$$a_r = \ddot{r} - r \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right)$$

$$= \ddot{r} - \frac{L^2}{m^2 r^3} \quad -(5)$$

so

$$\boxed{\ddot{r} = \frac{d^2 r}{dt^2} = a_r + \frac{L^2}{m^2 r^3}} \quad -(6)$$

The centrifugal acceleration is:

$$a_{cent} = \frac{L^2}{m^2 r^3}. \quad -(7)$$

Therefore:

$$a_\theta = \frac{1}{\sin\theta} \left(a_r \cos\theta - g \right) \quad -(8)$$

$$= \frac{a_r}{\tan\theta} - g$$

) and

$$a_\theta = \frac{1}{\tan\theta}\left(\ddot{r} - \frac{L^2}{m^2 r^3}\right) - g \quad -(9)$$

Fm Eq. (20) of Note 368(1) the force due to gravitation is counterbalanced by:

$$\underline{F} = \left(F_r \cos\theta - F_\theta \sin\theta\right)\underline{k} \quad -(10)$$

$$= m\left(\left(\ddot{r} - \frac{L^2}{m^2 r^3}\right)\cos\theta - a_\theta \sin\theta\right)\underline{k}$$

where

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2 \quad -(11)$$

$$= 2\dot{r}\dot{\theta} + r\ddot{\theta} - \frac{L_Z^2}{m^2 r^3}\frac{\cos\theta}{\sin^3\theta}$$

So:

$$\underline{F} = m\left(\left(\ddot{r} - \frac{L^2}{m^2 r^3}\right)\cos\theta - \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\sin\theta + \frac{L_Z^2}{m^2 r^3}\frac{\cos\theta}{\sin^2\theta}\right)\underline{k} \quad -(12)$$

$$= m\underline{g}$$

So:

$$g = \left(\ddot{r} - \frac{L^2}{m^2 r^3} \right) \cos\theta - \left(2\dot{r}\dot{\theta} + r\ddot{\theta} \right) \sin\theta + \frac{L_Z^2}{m^2 r^3} \frac{\cos\theta}{\sin^2\theta} \quad -(13)$$

in which L and L_Z are constants of motion.

As in UFT 270, if we define:

$$\dot{\beta}^2 = \dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \quad -(14)$$

the motion of the gyro is defined by:

$$\frac{d\beta}{dt} = \dot{\beta} = \frac{L}{mr^2} \quad -(15)$$

$$\frac{d\phi}{dt} = \frac{L_Z}{mr^2 \sin^2\theta} \quad -(16)$$

$$\frac{d\theta}{dt} = \frac{1}{mr^2} \left(L^2 - \frac{L_Z^2}{\sin^2\theta} \right)^{1/2} \quad -(17)$$

$$\frac{d\beta}{d\phi} = \frac{L}{L_Z} \sin^2\theta \quad -(18)$$

$$\beta = \int \frac{L \, d\theta}{\left(L^2 - \frac{L_Z^2}{\sin^2\theta} \right)^{1/2}} = -\sin^{-1} \left(\frac{L\cos\theta}{\left(L^2 - L_Z^2 \right)^{1/2}} \right), \quad -(18)$$

$$\cos\theta = -\frac{\left(L^2 - L_Z^2 \right)^{1/2}}{L} \sin\beta \quad -(19)$$

so

$$\beta = \tan^{-1} \left(\frac{L}{L_Z} \tan\phi \right) \quad -(20)$$

4)

$$\phi = \tan^{-1}\left(\frac{L_Z}{L}\tan\beta\right) \quad -(21)$$

and

$$\phi = -\frac{1}{2}\left(\sin^{-1}\left(\frac{(1+\cos\theta)L^2 - L_Z^2}{|1+\cos\theta|(L^4 - L_Z^2L^2)}\right)\right. \quad -(22)$$

$$\left. + \sin^{-1}\left(\frac{(\cos\theta - 1)L^2 + L_Z^2}{|(\cos\theta - 1)|(L^4 - L_Z^2L^2)}\right)\right)$$

These can be graphed as in UFT 270