365(1): Lagrangian Analysis with Spin Connections

In the non relativistic limit the hamiltonian is:

$$H = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + U(r) \quad -(1)$$

and the lagrangian is

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} - U(r) \quad -(2)$$

for a central potential. The three dimensional Euler Lagrange equation is:

$$\underline{\nabla} \mathcal{L} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} \quad -(3)$$

From eqs. (2) and (3):

$$\underline{F} = m \underline{\ddot{r}} = - \underline{\nabla} U \quad -(4)$$

Eq. (4) is true in any coordinate system.

The relativistic hamiltonian is:

$$H = mc^2 \left(1 - \frac{\underline{\dot{r}} \cdot \underline{\dot{r}}}{c^2} \right)^{-1/2} + U(r) \quad -(5)$$

and the relativistic lagrangian is:

$$\mathcal{L} = - mc^2 \left(1 - \frac{\underline{\dot{r}} \cdot \underline{\dot{r}}}{c^2} \right)^{1/2} - U(r) \quad -(6)$$

As shown in UFT 328, simultaneous solution of eqns. (5) and (6) gives a precessing orbit.

From eqs. (3) and (6):

$$\underline{F} = - \underline{\nabla} U = - \frac{d}{dt} \frac{\partial}{\partial \underline{\dot{r}}} \left(mc^2 \left(1 - \frac{\underline{\dot{r}} \cdot \underline{\dot{r}}}{c^2} \right)^{1/2} \right) \quad -(7)$$

a) Denote
$$y = 1 - \frac{\dot{\underline{r}} \cdot \dot{\underline{r}}}{c^2} \quad (8)$$
then
$$\frac{dy}{d\dot{\underline{r}}} = -\frac{2\dot{\underline{r}}}{c^2} \quad (9)$$
Denote:
$$f = y^{1/2} \quad (10)$$
then
$$\frac{df}{dy} = \frac{1}{2} y^{-1/2} \quad (11)$$
so
$$\frac{df}{d\dot{\underline{r}}} = \frac{df}{dy}\frac{dy}{d\dot{\underline{r}}} = -\frac{2\dot{\underline{r}}}{c^2}\frac{1}{2}y^{-1/2} \quad (12)$$
$$= -\frac{\dot{\underline{r}}}{c^2}\left(1 - \frac{\dot{\underline{r}} \cdot \dot{\underline{r}}}{c^2}\right)^{-1/2}$$
It follows that
$$\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} = -m\dot{\underline{r}}\left(1 - \frac{\dot{\underline{r}} \cdot \dot{\underline{r}}}{c^2}\right)^{-1/2} \quad (13)$$
and that:
$$\underline{F} = -m\frac{d}{dt}\left(\dot{\underline{r}}\left(1 - \frac{\dot{\underline{r}} \cdot \dot{\underline{r}}}{c^2}\right)^{-1/2}\right) \quad (14)$$
$$= -\underline{\nabla} U$$
In the non-relativistic limit:

3)

$$\underline{F} = m\ddot{\underline{r}} = m\frac{d\underline{v}}{dt} = -\underline{\nabla}U \quad -(15)$$

In plane polar coordinates:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta \quad -(16)$$

and

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(17)$$

$$= \frac{d\underline{r}}{dt}$$

The above analysis depends on:

$$\underline{r} = \underline{r}(t), \; \underline{v} = \underline{v}(t), \quad -(18)$$

so in plane polar coordinates:

$$\dot{\underline{r}} = \underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta. \quad -(19)$$

It follows that:

$$H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \quad -(20)$$

and the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \quad -(21)$$

This analysis produces the static ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(22)$$

and the results:

$$\dot{\theta} = \frac{L}{mr^2} \quad -(23)$$

and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(24)$$

4)

for any planar orbit. The force law for eq. (22) is therefore:

$$\underline{F} = m(\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r = -\underrightarrow{\nabla} U \quad -(25)$$

which is the 1689 Leibnitz equation. The centrifugal force is:

$$\underline{F}_{cent} = -mr\dot{\theta}^2\,\underline{e}_r \quad -(26)$$

The gradient of a function in plane polar coordinates is:

$$\underline{\nabla} U = \frac{\partial U}{\partial r}\,\underline{e}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\,\underline{e}_\theta \quad -(27)$$

so if

$$U = U(r) \quad -(28)$$

then

$$\underline{F} = m(\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r = -\frac{\partial U}{\partial r}\,\underline{e}_r \quad -(28)$$

so

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U(r)}{\partial r} = F(r) \quad -(29)$$

which can be transformed to the Binet equation:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(30)$$

Note carefully that the Binet equation (30) holds for any orbit in a plane, and is more general than a Newtonian analysis. It has not been assumed in Eqs. (29) and (30) that U(r) is an inverse square law.

In UFT36E3 it was shown that if the velocity is assumed to be:

5)

$$\underline{v} = \underline{v}(t, r(t), \theta(t)) \quad -(31)$$

then:

$$\underline{v} = \frac{D\underline{R}}{Dt} = \frac{\partial \underline{R}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{R} \quad -(32)$$

where $\underline{R}$ is the position vector of a fluid element:

$$\underline{R} = \underline{R}(t, r(t), \theta(t)) \quad -(33)$$

and

$$v_r = (1 + \Omega'_{01})\dot{r} + \Omega'_{02}\omega r \quad -(34)$$

$$v_\theta = \omega r = r\dot{\theta} \quad -(35)$$

where

$$\Omega'_{01} = \frac{\partial R}{\partial r}, \quad \Omega'_{02} = \frac{1}{r}\frac{\partial R}{\partial \theta} \quad -(36)$$

If it is assumed that:

$$\Omega'_{02} \sim 0 \quad -(37)$$

the Binet equation is changed to:

$$F(r) = -\frac{L^2}{mr^2}\left((1 + \Omega'_{01})\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(38)$$

Assume in this equation that:

$$F(r) = -\frac{mMG}{r^2} \quad -(39)$$

so

$$(1 + \Omega'_{01})\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{m^2 MG}{L^2} \quad -(40)$$

It is known that if:

6)

$$\Omega^1{}_{01} = 0 \quad -(41)$$

then:

$$\frac{m^2 MG}{L^2} = \frac{1}{\alpha} \quad -(42)$$

where

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(43)$$

when considering orbits in the solar system, in which precessions are very small, then:

$$\left(1+\Omega^1{}_{01}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha} \quad -(44)$$

This equation can be solved by computer to give r as a function of θ and $\Omega^1{}_{01}$. Under the condition:

$$v(t) \rightarrow v(t, r(t), \theta(t)) \quad -(45)$$

the inverse square law (39) no longer gives the ellipse (43). This is because the nature of spacetime is changed, and it has an effect on the orbit.

It is possible to find the value of $\Omega^1{}_{01}$ needed for a precessing ellipse. In a first approximation:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos(x\theta)\right) \quad -(46)$$

where

$$x = 1 - \frac{3MG}{c^2\alpha} \quad -(47)$$

So:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{x\epsilon}{\alpha}\sin(x\theta) \quad -(48)$$

and

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{x^2\epsilon}{\alpha}\cos(x\theta) \quad -(49)$$

Fm eqs. (44), (48) and (49):

$$-\left(1+\Omega'_{01}\right)\frac{x^2\epsilon}{\alpha}\cos(x\theta) + \frac{1}{\alpha}\left(1+\epsilon\cos(x\theta)\right) = \frac{1}{\alpha} \quad -(50)$$

i.e.

$$\frac{\epsilon}{\alpha}\cos(x\theta)\left(1 - x^2\left(1+\Omega'_{01}\right)\right) = 0 \quad -(51)$$

so

$$\boxed{x^2 = \frac{1}{1+\Omega'_{01}}} \quad -(52)$$

It follows that under the condition:

$$\boxed{\Omega'_{01} = \frac{1}{x^2} - 1 = \frac{\partial R}{\partial r}} \quad -(53)$$

the inverse square law (39) produces the function (43), which is a precessing ellipse.

More accurately the precessing ellipse is:

$$r = \frac{\alpha}{1+\epsilon\cos(f(\theta))} \quad -(54)$$

here $f(\theta)$ is a function of θ. So $f(\theta)$ can be found in terms of Ω'_{01} and vice versa, using computer algebra.