360(5): Expressions for Acceleration due to Gravity for Various Orbits

From the fundamental equation:

$$\underline{g} = (\underline{v} \cdot \underline{\nabla}) \underline{v} \quad -(1)$$

it follows the orbital velocity of any orbit in the XY plane is:

$$\underline{v} = (mG)^{1/2} \frac{(-X\underline{i} + Y\underline{j})}{(X^2+Y^2)^{3/4}} \quad -(2)$$

and that:

$$\underline{g} = -\frac{mG}{X^2+Y^2} \quad -(3)$$

is the Lagrange or convective derivative of $\underline{v}$, its derivative in the moving frame. Therefore:

$$v^2 = \frac{mG}{(X^2+Y^2)^{1/2}} = \dot{r}^2 + r^2\dot{\theta}^2 \quad -(4)$$

For an elliptical orbit the observed orbital velocity is:

$$v^2 = mG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(5)$$

where a is the semi major axis of the ellipse,

so

$$\frac{1}{(X^2+Y^2)^{1/2}} = \frac{2}{r} - \frac{1}{a} = \frac{1}{mG}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad -(6)$$

Therefore:

2)

$$\underset{\sim}{g} = -\frac{mG}{x^2+y^2} e_r = -mG\left(\frac{2}{r} - \frac{1}{a}\right)^2 \underline{e}_r \quad -(7)$$
$$= -\frac{1}{mG}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)^2 \underline{e}_r$$

Note carefully that this acceleration due to gravity is defined by an inverse square law but it is also defined by a convective derivative, so it is not the Newtonian definition, it is a generally covariant definition:

$$\underline{g} = (\underline{v}\cdot\underline{\nabla})\underline{v} = -\frac{\partial \underline{v}}{\partial t} - \underline{\nabla}\phi \quad -(8)$$

where $\underline{v}$ plays the role of vector potential and ϕ plays the role of scalar potential:

$$\phi = h \quad -(9)$$

where h is enthalpy.

In electrodynamics:

$$\underline{E} = x(\underline{A}\cdot\underline{\nabla})\underline{A} = -\frac{\partial A}{\partial t} - \underline{\nabla}\phi_e \quad -(10)$$

and in ECE2:

$$\underline{E} = x(\underline{W}\cdot\underline{\nabla})\underline{W} = -\frac{\partial \underline{W}}{\partial t} - \underline{\nabla}\phi_W \quad -(11)$$

where x is a constant.

Therefore eq. (7) is a completely new kind of inverse square law, the acceleration due to gravity is the convective derivative of the orbital velocity:

$$\boxed{\underline{g} = -\frac{v^4}{mG}\,\underline{e}_r} \quad -(12)$$

In the Newtonian theory:

$$\underline{g} = -\underline{\nabla}\phi \quad -(13)$$

and

$$H = \frac{1}{2}mv^2 - \underline{\nabla}\phi \quad -(14)$$

where $\underline{v}$ is the orbital velocity. The Newtonian theory uses:

$$\phi = -\frac{mG}{r}, \quad -(15)$$

the Newtonian gravitational potential. So:

$$\underline{g_N} = -\frac{mG}{r^2}\,\underline{e}_r \quad -(16)$$

is the Newtonian acceleration due to gravitation.

Eq (7) reduces to the Newtonian theory for a circular orbit:

$$r = a \quad -(17)$$

but the relations present in eq. (7) are missing completely from the Newtonian theory. Eq. (16) is galilean covariant but eq. (7) is generally covariant, i.e. is an equation of general relativity.

The Newton theory applies only to conic sections such the ellipse but fluid gravitation applies to all orbits, including three dimensional orbits, via eq. (1).

For example in a whirlpool galaxy:

4) $$v^2 = \frac{L^2}{m^2}\left(\frac{1}{r^2} + \frac{1}{r_0^2}\right) \quad -(18)$$

where
$$r = \frac{r_0}{\theta} \quad -(19)$$

is the hyperbolic spiral orbit. Here L is the constant angular momentum, a constant of motion, and m is the mass of an orbiting star. Therefore:

$$\underline{g} = -\frac{v^4}{MG}\,\underline{e}_r = -\frac{L^4}{m^4 MG}\left(\frac{1}{r^2} + \frac{1}{r_0^2}\right)^2 \underline{e}_r$$

i.e.

$$\boxed{\underline{g} = -\frac{1}{MG}\left(\frac{L^2}{m^2}\left(\frac{1}{r^2} + \frac{1}{r_0^2}\right)\right)^2 \underline{e}_r} \quad -(20) \quad -(21)$$

In a precessing elliptical orbit, as in the previous note:

$$\dot{r} = \frac{dr}{dt} = \frac{x\epsilon L}{m\alpha}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad -(22)$$

and

$$\dot{\theta} = \frac{L}{mr^2} \quad -(23)$$

so:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \frac{L^2}{m^2}\left(\frac{x^2\epsilon^2}{\alpha^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right) + \frac{1}{r^2}\right) \quad -(24)$$

5) So the force law for a precessing elliptical orbit in a plane is:

$$\underline{F} = m\underline{g} = -\frac{mv^4}{MG}\underline{e}_r \quad -(25)$$

with v^2 given by Eq. (24). This force law is of the form:

$$\underline{F} = -\frac{mMG}{x^2+y^2}\underline{e}_r \quad -(26)$$

and is the inverse square law of general relativity.