360(4): Force Law for a Precessing Ellipse

The astronomically observed orbit of a planet of mass m about the sun of mass M is the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(1)$$

where

$$x = 1 + \frac{3MG}{c^2 a(1-\epsilon^2)} \quad -(2)$$

to high experimental precision.

The semi major axis a is defined by:

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(3)$$

so

$$x = 1 + \frac{3MG}{c^2 \alpha} \quad -(4)$$

Here α is the half right latitude and ϵ is the eccentricity. The quantity:

$$x = 1 + \frac{3MG}{c^2 \alpha} \quad -(5)$$

is very close to unity in the solar system.

The factor x in Eq. (5) is considered to be experimentally observed.

In plane polar coordinates the Lagrangian of the precessing ellipse is:

2) $$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \quad - (6)$$

where $U(r)$ is the potential energy. Here:

$$\mu = \frac{mM}{m+M} \sim m \quad - (7)$$

From the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \quad - (8)$$

it is found that the angular momentum L is a constant of motion, so:

$$L = mr^2\frac{d\theta}{dt} = \text{constant} \quad - (9)$$

Therefore

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{L}{mr^2} \quad - (10)$$

Also:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{L}{mr^2}\frac{dr}{d\theta} \quad - (11)$$

From eq. (1):

$$\frac{dr}{d\theta} = \frac{x\epsilon r^2}{\alpha}\sin(x\theta) \quad - (12)$$

So

$$\frac{dr}{dt} = \frac{x\epsilon L}{m\alpha}\sin(x\theta) \quad - (13)$$

Using:

$$\cos^2(x\theta) + \sin^2(x\theta) = 1 \quad -(14)$$

it follows that:

$$\sin(x\theta) = \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad -(15)$$

So

$$\dot{r} = \frac{dr}{dt} = \frac{x\epsilon L}{m\alpha}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad -(16)$$

$$\dot{\theta} = \frac{L}{mr^2} \quad -(17)$$

Using Eqs. (16) and (17), X and Y can be worked out as follows:

$$X = mG\left(\frac{\dot{r}\sin\theta + r\dot{\theta}\cos\theta}{(\dot{r}^2 + r^2\dot{\theta}^2)^{3/2}}\right), \quad -(17)$$

$$Y = mG\left(\frac{r\dot{\theta}\sin\theta - \dot{r}\cos\theta}{(\dot{r}^2 + r^2\dot{\theta}^2)^{3/2}}\right) \quad -(18)$$

in terms of r and θ.

This defines a moving coordinate system.

In fluid gravitation, the force law needed for a precessing elliptical orbit is the Lagrange or convective derivative of the orbital velocity $\underline{v}$.

In general:

$$\underline{F} = m\underline{g} = m(\underline{v}\cdot\underline{\nabla})\underline{v}$$

$$= -\frac{mMG}{r^2}\,\underline{e}_r \quad -(19)$$

Note carefully that this is true for any orbit, in three dimensions. The Lagrange derivative of the orbital linear velocity is the derivative in the moving reference frame or coordinate system. The coordinate system is defined by the astronomical observations of the orbit.

Therefore the Hooke / Newton inverse square law has been generalized to any orbit.

In two dimensions:

$$\underline{F} = -\frac{mMG}{X^2+Y^2}\,\underline{e}_r \quad -(20)$$

so

$$\underline{g} = -\frac{MG}{X^2+Y^2}\,\underline{e}_r \quad -(21)$$

and a solution of eq. (19) is:

$$\underline{v} = (MG)^{1/2}\,\frac{(-Y\underline{i} + X\underline{j})}{(X^2+Y^2)^{3/4}} \quad -(22)$$

In plane polar coordinates:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(23)$$

where

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(24)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(25)$$

Therefore the orbital velocity for any orbit, in ~~dimension~~, is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta = (MG)^{1/2}\frac{(-Y\underline{i} + X\underline{j})}{(X^2+Y^2)^{3/4}} \quad -(26)$$

and eqs. (17) and (18) follow.