...o(3): The Force Law for a Whirlpool Galaxy

In this case the general law of fluid gravitation:

$$\underline{F} = m\underline{g} = m(\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(1)$$

reduces to:

$$\underline{F} = m\underline{g} = -\frac{mM}{r^2}\,\underline{e}_r \quad -(2)$$

where

$$r^2 = X^2 + Y^2 \quad -(3)$$

and

$$X = mG\left(\frac{\dot{r}\sin\theta + r\dot{\theta}\cos\theta}{(\dot{r}^2 + r^2\dot{\theta}^2)^{3/2}}\right) \quad -(4)$$

$$Y = mG\left(\frac{r\dot{\theta}\sin\theta - \dot{r}\cos\theta}{(\dot{r}^2 + r^2\dot{\theta}^2)^{3/2}}\right) \quad -(5)$$

in plane polar coordinates (r, θ). For a whirlpool galaxy the orbit is a hyperbolic spiral:

$$r = \frac{r_0}{\theta} \quad -(6)$$

and

$$\dot{r}^2 + r^2\dot{\theta}^2 = \frac{L^2}{m^2}\left(\frac{1}{r^2} + \frac{1}{r_0^2}\right) \quad -(7)$$

where L is the angular momentum, a constant of motion, and m is the mass of a star orbiting a central galactic mass M. From Lagrangian dynamics:

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{L}{mr^2}. \quad -(8)$$

So:

a)

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{L}{mr^2}\frac{dr}{d\theta} \quad -(9)$$

Fm eq. (6):

$$\frac{dr}{d\theta} = -\frac{r_0}{\theta^2} = -\frac{r^2}{r_0} \quad -(10)$$

Therefore:

$$\dot{r} = \frac{dr}{dt} = -\frac{L}{mr_0} \quad -(11)$$

$$= \text{constant}$$

Finally:

$$\cos\theta = \cos\left(\frac{r_0}{r}\right) \; ; \; \sin\theta = \left(\frac{r_0}{r}\right) \quad -(12)$$

in the range of r, which is:

$$0 < r < \infty. \quad -(13)$$

So:

$$X = mG\frac{\left(\frac{L}{mr}\cos\left(\frac{r_0}{r}\right) - \frac{L}{mr_0}\sin\left(\frac{r_0}{r}\right)\right)}{\left(\frac{L^2}{m^2}\left(\frac{1}{r^2}+\frac{1}{r_0^2}\right)\right)^{3/2}} \quad -(14)$$

and

$$Y = mG\frac{\left(\frac{L}{mr}\sin\left(\frac{r_0}{r}\right) - \frac{L}{mr_0}\cos\left(\frac{r_0}{r}\right)\right)}{\left(\frac{L^2}{m^2}\left(\frac{1}{r^2}+\frac{1}{r_0^2}\right)\right)^{3/2}} \quad -(15)$$

It follows that the hyperbolic spiral orbit can be produced by an inverse square law given the fundamental force law (1) of fluid gravitation, quod erat demonstrandum.

Furthermore, the generally covariant law (1) describes all planar orbits. They are all described by an inverse square law (2). The orbit itself is found by astronomical observation.

Finally:

$$X \xrightarrow[r \to \infty]{} 0 \quad -(14)$$

$$Y \xrightarrow[r \to \infty]{} MG\left(\frac{m r_0}{L}\right)^2 \quad -(15)$$