366(1): Self Consistency Check, Newtonian Field

Given:

$$x = \alpha^{1/2} \frac{\left(\frac{\epsilon}{\alpha}\sin^2\theta + \frac{1}{r}\cos\theta\right)}{\left(\frac{2}{r} - \frac{1}{a}\right)^{3/2}} \quad -(1)$$

$$y = \alpha^{1/2} \frac{\left(\frac{1}{r}\sin\theta - \frac{\epsilon \sin\theta\cos\theta}{\alpha}\right)}{\left(\frac{2}{r} - \frac{1}{a}\right)^{3/2}} \quad -(2)$$

it must be shown that

$$\frac{1}{(x^2+y^2)^{1/2}} = \frac{2}{r} - \frac{1}{a} \quad -(3)$$

From eqns (1) and (2):

$$x^2 + y^2 = \frac{\alpha}{\left(\frac{2}{r} - \frac{1}{a}\right)^3}\left(\left(\frac{\epsilon}{\alpha}\sin^2\theta + \frac{1}{r}\cos\theta\right)^2 + \left(\frac{1}{r}\sin\theta - \frac{\epsilon\sin\theta\cos\theta}{\alpha}\right)^2\right)$$

$$= \frac{\alpha}{\left(\frac{2}{r} - \frac{1}{a}\right)^3}\left(\left(\frac{\epsilon}{\alpha}\right)^2\sin^4\theta + \frac{2\epsilon}{\alpha r}\sin^2\cos\theta + \frac{1}{r^2}\cos^2\theta + \frac{1}{r^2}\sin^2\theta - \frac{2\epsilon}{r\alpha}\sin^2\theta\cos\theta + \left(\frac{\epsilon}{\alpha}\right)^2\sin^2\theta\cos^2\theta\right)$$

$$= \frac{\alpha}{\left(\frac{2}{r} - \frac{1}{a}\right)^3}\left(\frac{1}{r^2} + \left(\frac{\epsilon}{\alpha}\right)^2\sin^2\theta\right) \quad -(4)$$

Now use $\quad \frac{1}{r^2} = \frac{1}{\alpha^2}\left(1 + \epsilon\cos\theta\right)^2 \quad -(5)$

So:

$$x^2+y^2 = \frac{\alpha}{\left(\frac{2}{r}-\frac{1}{a}\right)^3}\left(1+2\epsilon\cos\theta+\epsilon^2\right)$$

$$= \frac{1}{\alpha\left(\frac{2}{r}-\frac{1}{a}\right)^3}\left(1+2\left(\frac{\alpha}{r}-1\right)+\epsilon^2\right)$$

$$= \left(\frac{2}{r}+\frac{\epsilon^2-1}{\alpha}\right)\Big/\left(\frac{2}{r}-\frac{1}{a}\right)^3$$

$$= \left(\frac{2}{r}-\frac{1}{a}\right)\Big/\left(\frac{2}{r}-\frac{1}{a}\right)^3$$

$$= \frac{1}{\left(\frac{2}{r}-\frac{1}{a}\right)^2} \quad -(6)$$

So

$$\frac{1}{(x^2+y^2)^{1/2}} = \frac{2}{r}-\frac{1}{a} \quad -(7)$$

Q.E.D.

In general, for any planar orbit:

$$\underline{g} = (\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(8)$$

also

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(9)$$

The Newtonian orbit is:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(10)$$

), but the experimentally observed orbit is a precessing ellipse. For the precessing ellipse:

$$\underline{v} = (MG)^{1/2} \frac{\left(-Y\underline{i} + X\underline{j}\right)}{\left(X^2+Y^2\right)^{3/4}} \quad -(11)$$

so X and Y must be found for the precessing ellipse in terms of r and θ. Having done this, the force law for the observed orbit can be found using eq. (8):

$$\boxed{\underline{F} = m\underline{g} = m(\underline{v}\cdot\underline{\nabla})\underline{v}} \quad -(12)$$

where $\underline{v}$ is the orbital velocity observed experimentally.

It is known experimentally that the perihelion is displaced in every revolution by:

$$\Delta = \frac{6\pi MG}{ac^2(1-\epsilon^2)} \quad -(13)$$

i.e.

$$\theta = 2\pi \rightarrow 2\pi\left(1 + \frac{3MG}{ac^2(1-\epsilon^2)}\right) \quad -(14)$$

$$= x\theta$$

So

$$\boxed{x = 1 + \frac{3MG}{ac^2(1-\epsilon^2)}} \; ; \text{ so } \boxed{r = \frac{\alpha}{1+\epsilon\cos(x\theta)}} \quad -(15)$$